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M/G/1 QUEUE WITH COMPULSORY VACATION AND THREE PHASE REPAIRS

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ABSTRACT

We study a single server M/G/1 queueing system with Poisson input subject to compulsory server vacation and random breakdowns. The server provides service to all arriving customers on a first come first served basis and service times follow general distribution. However after the completion of each service the server will take compulsory vacation. The system may breakdown at random and it must be send to repair process immediately. Further the repair process involves three phases of repairs with different general repair time distributions. The supplementary variable technique is applied to find explicitly probability generating function of the number in the system

Key words: Poisson arrivals, general times, probability generating function, random breakdowns, idle state, steady state, deterministic repairs, supplementary variable technique.

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1. INTRODUCTION

Single server queueing systems subject to vacations or interruptions have been studied by numerous researchers due to their wide applications in the analysis of processor schedules, the analysis of polling systems, analysis of manufacturing systems and computer communication systems. Vacation may take place when the human server in the system wishes to take a pause or leave the system for a certain period of time. Vacation queues have been studied by several authors including Doshi [1,2], Takagi [10], Levy and Yechiali [4], Madan [5,6,7,9], Thangaraj and Vanitha [11].

In real life situations, a queueing system might suddenly breakdown and hence the server will not be able to provide service unless the system is repaired. Madan and Maraghi [8] have studied batch arrival queueing system with random breakdowns and Bernoulli schedule server vacation having general vacation time. Vanitha [12] analyzed a queueing system with deterministic repair times. The most realistic aspect in modeling of an unreliable server is phase type of repairs. Hsieh et al. [3] studied a queueing model in which the server is subject to several types of breakdowns and each type has two possible stages of repair.

In this paper we consider a single server queueing model subject to compulsory vacation and breakdowns. As soon as the customer's service is complete the server will go for compulsory vacation. The system may breakdown at random with breakdown rate $\alpha > 0$. As soon as the system is broken down it is immediately sent for repair where the repair is provided in three phases. After first stage of repair, every failed unit has to undergo second phase of repair and then the third phase of repair. After completion of third phase of repair, the server resumes its work immediately. Also the customer whose service is interrupted comes back to the head of the queue.

The rest of the paper is organized as follows. The mathematical description of our model is in section 2 and the equations governing the model are given in section 3. The time dependent solution have been obtained in section 4 using supplementary variable technique and the corresponding steady state results have been derived explicitly in section 5.

2. ASSUMPTIONS UNDERLYING THE MODEL

The following assumptions describe the mathematical model

- Customers arrive at the system one by one in according to a Poisson
- stream with arrival rate $\lambda > 0$.
- The service to customers is provided one by one on a first come, first served
- basis and their service times follow a general (arbitrary) distribution with distribution function $B(v)$ and the density function $b(v)$.
- Let $\mu(x)dx$ be the conditional probability of completion of a service during the interval $(x + dx]$ given that elapsed time is x, so that

$$\mu(\mathrm{x}) = \frac{b(x)}{1-B(x)} \tag{2.1}$$

and therefore

$$b(v) = \mu(v)e^{-\int_0^v \mu(x)dx} \ . \tag{2.2}$$

- As soon as the service of a customer is complete, the server will take a compulsory vacation of random length. The vacation time follow general (arbitrary) distribution with distribution $V(s)$ and the density function $v(s)$. Let $\gamma(x)dx$ be the conditional probability of a completion of a vacation during the interval $(x + dx]$ given that the elapsed vacation time is x, so that

$$\gamma(x) = \frac{v(x)}{1-V(x)} \tag{2.3}$$

and therefore,

$$v(s) = \gamma(s)e^{-\int_0^s \gamma(s)ds} \ . \tag{2.4}$$

- On returning from vacation the server instantly starts serving the customer at the head of the queue, if any.
- The customers are served according to the first come, first served rule.

- The server is subject to random breakdowns such that αdt is the first order probability that the service channel will fail during the short interval of time $(t, t + dt]$.
- We assume that whenever service channel breaks down, it instantly undergoes a repair process and the customer whose service gets interrupted comes back to the head of the queue.
- We assume that the repair process comprises of three phases of repairs, the first phase followed by the second phase and then the third phase.
- The duration of repairs follow a general (arbitrary) distribution with distribution functions $H(r_i)$ and density functions $h(r_i)$, $i = 1,2,3$.
- Let $\beta_i(x)dx$ be the conditional probability of a completion of i^{th} stage of repair during the interval $(x + dx]$ given that elapsed repair time is x, so that

$$\beta_i(\mathrm{x}) = \frac{r_i(x)}{1-R_i(x)} \qquad (2.3)$$

and therefore

$$r_i(t) = \beta_i(t)e^{-\int_0^t \beta(x)dx}, i = 1,2,3 . \qquad (2.4)$$

- Various Stochastic Processes involved in the system are independent of each other.

3. DEFINITIONS, NOTATIONS AND THE TIME – DEPENDENT EQUATIONS GOVERNING THE SYSTEM

We define

- $P_n(x, t)$: Probability that at time t, there are $n \geq 0$ customers in the queue excluding one customer in service and the elapsed served time of this customer is x. Consequently $P_n(t) = \int_0^\infty P_n(x, t)dx$ denotes the probability that at time t, there are n customers in the queue excluding the one customer in service irrespective of the value of x .
- $V_n(x, t)$: Probability that at time t, the server is under vacation with elapsed vacation time x and there are $n \geq 0$ customers waiting in the queue for service. Consequently $V_n(t) = \int_0^\infty V_n(x, t)dx$ denotes the probability that at time t, there are n customers in the queue and the server is under vacation irrespective of the value of x .
- $R_n^{(1)}(x, t)$: Probability that at time t, the server is inactive due to system breakdown and the server is under first phase of repair with elapsed repair time x and there are n customers waiting in the queue for repair. Consequently $R_n^{(1)}(t) = \int_0^\infty R_n^{(1)}(x, t)dx$ denotes the probability that at time t, there are n customers in the queue and the server is under first phase of repair irrespective of the value of x .
- $R_n^{(2)}(x, t)$: Probability that at time t, the server is inactive due to system breakdown and the server is under second phase of repair with elapsed repair time x and there are n customers waiting in the queue for repair. Consequently $R_n^{(2)}(t) = \int_0^\infty R_n^{(2)}(x, t)dx$ denotes the probability that at time t, there are n customers in the queue and the server is under second phase of repair irrespective of the value of x .

- $R_n^{(3)}(x,t)$: Probability that at time t, the server is inactive due to system breakdown and the server is under third phase of repair with elapsed repair time x and there are n customers waiting in the queue for repair. Consequently $R_n^{(3)}(t) = \int_0^\infty R_n^{(3)}(x,t)dx$ denotes the probability that at time t, there are n customers in the queue and the server is under third phase of repair irrespective of the value of x .
- $Q_n(t)$: Probability that at time t, there is no customer in the system and the server is idle but available in the system.

The model is then governed by the following time dependent forward system equations

$$\frac{\partial}{\partial x}P_n(x,t) + \frac{\partial}{\partial t}P_n(x,t) + (\lambda + \mu(x) + \alpha)P_n(x,t) = \lambda P_{n-1}(x,t), n = 1,2, ... \quad (3.1)$$

$$\frac{\partial}{\partial x}P_0(x,t) + \frac{\partial}{\partial t}P_0(x,t) + (\lambda + \mu(x) + \alpha)P_0(x,t) = 0, \quad (3.2)$$

$$\frac{\partial}{\partial x}V_n(x,t) + \frac{\partial}{\partial t}V_n(x,t) + \big(\lambda + \gamma(x)\big)V_n(x,t) = \lambda V_{n-1}(x,t), n = 1,2, ... \quad (3.3)$$

$$\frac{\partial}{\partial x}V_0(x,t) + \frac{\partial}{\partial t}V_0(x,t) + \big(\lambda + \gamma(x)\big)V_0(x,t) = 0, \quad (3.4)$$

$$\frac{\partial}{\partial x}R_n^{(1)}(x,t) + \frac{\partial}{\partial t}R_n^{(1)}(x,t) + \big(\lambda + \beta_1(x)\big)R_n^{(1)}(x,t) = \lambda R_{n-1}^{(1)}(x,t), n = 1,2, \quad (3.5)$$

$$\frac{\partial}{\partial x}R_0^{(1)}(x,t) + \frac{\partial}{\partial t}R_0^{(1)}(x,t) + \big(\lambda + \beta_1(x)\big)R_0^{(1)}(x,t) = 0, \quad (3.6)$$

$$\frac{\partial}{\partial x}R_n^{(2)}(x,t) + \frac{\partial}{\partial t}R_n^{(2)}(x,t) + \big(\lambda + \beta_2(x)\big)R_n^{(2)}(x,t) = \lambda R_{n-1}^{(2)}(x,t), n = 1,2, ... \quad (3.7)$$

$$\frac{\partial}{\partial x}R_0^{(2)}(x,t) + \frac{\partial}{\partial t}R_0^{(2)}(x,t) + \big(\lambda + \beta_2(x)\big)R_0^{(2)}(x,t) = 0, \quad (3.8)$$

$$\frac{\partial}{\partial x}R_n^{(3)}(x,t) + \frac{\partial}{\partial t}R_n^{(3)}(x,t) + \big(\lambda + \beta_3(x)\big)R_n^{(3)}(x,t) = \lambda R_{n-1}^{(3)}(x,t), n = 1,2, ... \quad (3.9)$$

$$\frac{\partial}{\partial x}R_0^{(3)}(x,t) + \frac{\partial}{\partial t}R_0^{(3)}(x,t) + \big(\lambda + \beta_3(x)\big)R_0^{(3)}(x,t) = 0, \quad (3.10)$$

$$\frac{d}{dt}Q(t) = -\lambda Q(t) + \int_0^\infty R_0^{(3)}(x,t)\beta_3(x)dx + \int_0^\infty V_0(x,t)\gamma(x)dx. \quad (3.11)$$

Equations (3.1) to (3.11) are to be solved subject to the following boundary conditions:

$$P_0(0,t) = \lambda Q(t) + \int_0^\infty R_1^{(3)}(x,t)\beta_3(x)dx + \int_0^\infty V_1(x,t)\gamma(x)dx, \quad (3.12)$$

$$P_n(0,t) = \int_0^\infty R_{n+1}^{(3)}(x,t)\beta_3(x)dx + \int_0^\infty V_{n+1}(x,t)\gamma(x)dx, \ n = 1,2, ... \quad (3.13)$$

$$V_n(0,t) = \int_0^\infty P_n(x,t)\mu(x)dx, n = 0,1, ..., \quad (3.14)$$

$$R_0^{(1)}(0,t) = 0 \quad , \quad (3.15)$$

$$R_n^{(1)}(0,t) = \alpha\int_0^\infty P_{n-1}(x,t)dx, \quad n = 1,2, ..., \quad (3.16)$$

$$R_n^{(2)}(0,t) = \int_0^\infty R_n^{(1)}(x,t)\beta_1(x)dx, \quad n = 0,1,\dots, \tag{3.17}$$

$$R_n^{(3)}(0,t) = \int_0^\infty R_n^{(2)}(x,t)\beta_2(x)dx, \quad n = 0,1,\dots, \tag{3.18}$$

Next, we assume that initially the server is available but idle because of no customers so that the initial condition is

$$Q(0) = 1. \tag{3.19}$$

4. GENERATING FUNCTIONS OF THE QUEUE LENGTH: THE TIME DEPENDENT SOLUTION

In this section we define the transient solution for the above set of differential-difference equations. We define the probability generating functions

$$\left.\begin{aligned}
P(x,z,t) &= \sum_{n=0}^\infty z^n P(x,t), P(z,t) = \sum_{n=0}^\infty z^n P(t),\\
V(x,z,t) &= \sum_{n=0}^\infty z^n V(x,t), V(z,t) = \sum_{n=0}^\infty z^n V(t),\\
R^{(1)}(x,z,t) &= \sum_{n=0}^\infty z^n R^{(1)}(x,t), R^{(1)}(z,t) = \sum_{n=0}^\infty z^n R^{(1)}(t),\\
R^{(2)}(x,z,t) &= \sum_{n=0}^\infty z^n R^{(2)}(x,t), R^{(2)}(z,t) = \sum_{n=0}^\infty z^n R^{(2)}(t),\\
R^{(3)}(x,z,t) &= \sum_{n=0}^\infty z^n R^{(3)}(x,t), R^{(3)}(z,t) = \sum_{n=0}^\infty z^n R^{(3)}(t).
\end{aligned}\right\} \tag{4.1}$$

which are convergent inside the circle given by $|z| \le 1$and define the Laplace transform of a function $f(t)$ as

$$\overline{f(s)} = \int_0^\infty e^{-st} f(t)\,dt, \Re(s) > 0\ . \tag{4.2}$$

We take Laplace transforms of equations (3.1) to (3.18) and using (3.19), we obtain

$$\frac{\partial}{\partial x}\overline{P_n}(x,s) + (s+\lambda+\mu(x)+\alpha)\overline{P_n}(x,s) = \lambda\overline{P}_{n-1}(x,s), n = 1,2,\dots, \tag{4.3}$$

$$\frac{\partial}{\partial x}\overline{P_0}(x,s) + (s+\lambda+\mu(x)+\alpha)\overline{P_0}(x,s) = 0, \tag{4.4}$$

$$\frac{\partial}{\partial x}\overline{V_n}(x,s) + \big(s+\lambda+\gamma(x)\big)\overline{V_n}(x,s) = \lambda\overline{V}_{n-1}(x,s), n = 1,2,\dots, \tag{4.5}$$

$$\frac{\partial}{\partial x}\overline{V_0}(x,s) + \big(s+\lambda+\gamma(x)\big)\overline{V_0}(x,s) = 0, \tag{4.6}$$

$$\frac{\partial}{\partial x}\overline{R_n}^{(1)}(x,s) + \big(s+\lambda+\beta_1(x)\big)\overline{R_n}^{(1)}(x,s) = \lambda\overline{R}_{n-1}^{(1)}(x,s),$$
$$n = 1,2,\dots \tag{4.7}$$

$$\frac{\partial}{\partial x}\overline{R_0}^{(1)}(x,s) + \big(s+\lambda+\beta_1(x)\big)\overline{R_0}^{(1)}(x,s) = 0, \tag{4.8}$$

$$\frac{\partial}{\partial x}\overline{R_n}^{(2)}(x,s) + \big(s+\lambda+\beta_2(x)\big)\overline{R_n}^{(2)}(x,s) = \lambda\overline{R}_{n-1}^{(2)}(x,s),$$
$$n = 1,2,\dots \tag{4.9}$$

$$\frac{\partial}{\partial x}\overline{R_0}^{(2)}(x,s) + \big(s+\lambda+\beta_2(x)\big)\overline{R_0}^{(2)}(x,s) = 0, \tag{4.10}$$

$$\frac{\partial}{\partial x}\overline{R_n}^{(3)}(x,s) + \big(s+\lambda+\beta_3(x)\big)\overline{R_n}^{(3)}(x,s) = \lambda\overline{R}_{n-1}^{(3)}(x,s),$$
$$n = 1,2,\dots \tag{4.11}$$

$$\frac{\partial}{\partial x}\overline{R_0}^{(3)}(x,s) + \big(s+\lambda+\beta_3(x)\big)\overline{R_0}^{(3)}(x,s) = 0, \tag{4.12}$$

$$(s+\lambda)Q(s) = 1 + \int_0^\infty \overline{R}_O^{(3)}(x,s)\beta_3(x)dx + \int_0^\infty \overline{V}_0(x,s)\gamma(x)dx, \tag{4.13}$$

$$\overline{P}_0(0,s) = \lambda\overline{Q}(s) + \int_0^\infty \overline{R}_1^{(3)}(x,s)\beta_3(x)dx + \int_0^\infty \overline{V}_1(x,s)\gamma(x)\,dx, \tag{4.14}$$

$$\overline{P}_n(0,s) = \int_0^\infty \overline{R}^{(3)}_{n+1}(x,s)\beta_3(x)dx + \int_0^\infty \overline{V}_{n+1}(x,s)\gamma(x)dx \,, n = 1,2, \ldots \quad (4.15)$$

$$\overline{V_n}(0,s) = \int_0^\infty \overline{P_n}(x,s)\mu(x)dx, n = 0,1, \ldots, . \quad (4.16)$$

$$\overline{R_n}^{(1)}(0,s) = 0 \,, \quad (4.17)$$

$$\overline{R_n}^{(1)}(0,s) = \alpha \int_0^\infty P_{n-1}(x,s)\,dx, \quad n = 1,2, \ldots, \quad (4.18)$$

$$\overline{R_n}^{(2)}(0,s) = \int_0^\infty R_n^{(1)}(x,s)\beta_1(x)dx, \quad n = 0,1, \ldots, \quad (4.19)$$

$$\overline{R_n}^{(3)}(0,s) = \int_0^\infty R_n^{(2)}(x,s)\beta_2(x)dx, \quad n = 0,1, \ldots, \quad (4.20)$$

We multiply equations (4.3) – (4.12) and (4.14) – (4.20) by suitable powers of z, using equations (4.1), (3.11) and then simplify we obtain

$$\frac{\partial}{\partial x}\overline{P}(x,z,s) + (s + \lambda - \lambda z + \mu(x) + \alpha)\overline{P}(x,z,s) = 0, \quad (4.21)$$

$$\frac{\partial}{\partial x}\overline{V}(x,z,s) + (s + \lambda - \lambda z + \gamma(x))\overline{V}(x,z,s) = 0, \quad (4.22)$$

$$\frac{\partial}{\partial x}\overline{R}^{(1)}(x,z,s) + (s + \lambda - \lambda z + \beta_1(x))\overline{R}^{(1)}(x,z,s) = 0, \quad (4.23)$$

$$\frac{\partial}{\partial x}\overline{R}^{(2)}(x,z,s) + (s + \lambda - \lambda z + \beta_2(x))\overline{R}^{(2)}(x,z,s) = 0, \quad (4.24)$$

$$\frac{\partial}{\partial x}\overline{R}^{(3)}(x,z,s) + (s + \lambda - \lambda z + \beta_3(x))\overline{R}^{(3)}(x,z,s) = 0, \quad (4.25)$$

$$z\,\overline{P}(0,z,s) = (1 - s\overline{Q}(s)) + \lambda(z-1)\overline{Q}(s) + \int_0^\infty \overline{R}^{(3)}(x,z,s)\beta_3(x)dx +$$

$$-\int_0^\infty \overline{V}(x,z,s)\gamma(x)dx, \quad (4.26)$$

$$\overline{V}(0,z,s) = \int_0^\infty \overline{P}(x,z,s)\mu(x)dx, \quad (4.27)$$

$$\overline{R}^{(1)}(0,z,s) = \alpha z \int_0^\infty \overline{P}(x,z,s)dx, \quad (4.28)$$

$$\overline{R}^{(2)}(0,z,s) = \int_0^\infty \overline{R}^{(1)}(x,z,s)\beta_1(x)dx, \quad (4.29)$$

$$\overline{R}^{(3)}(0,z,s) = \int_0^\infty \overline{R}^{(2)}(x,z,s)\beta_2(x)dx, \quad (4.30)$$

Next we integrate equations (4.21) – (4.25) between limits 0 and x and obtain

$$\overline{P}(x,z,s) = \overline{P}(0,z,s)e^{-(s+\lambda-\lambda z+\alpha)x-\int_0^x \mu(t)dt}, \quad (4.31)$$

$$\overline{V}(x,z,s) = \overline{V}(0,z,s)e^{-(s+\lambda-\lambda z)x-\int_0^x \gamma(t)dt}, \quad (4.32)$$

$$\overline{R}^{(1)}(x,z,s) = \overline{R}^{(1)}(0,z,s)e^{-(s+\lambda-\lambda z)x-\int_0^x \beta_1(t)dt}, \quad (4.33)$$

$$\overline{R}^{(2)}(x,z,s) = \overline{R}^{(2)}(0,z,s)e^{-(s+\lambda-\lambda z)x-\int_0^x \beta_2(t)dt}, \quad (4.34)$$

$$\overline{R}^{(3)}(x,z,s) = \overline{R}^{(3)}(0,z,s)e^{-(s+\lambda-\lambda z)x-\int_0^x \beta_3(t)dt}, \quad (4.35)$$

We again integrate equations (4.31) – (4.35) with respect to x by parts

$$\overline{P}(z,s) = \overline{P}(0,z,s)\left[\frac{1-\overline{B}(S+\lambda-\lambda z+\alpha)}{s+\lambda-\lambda z+\alpha}\right] \quad (4.36)$$

Where $\overline{B}(s + \lambda - \lambda z + \alpha) = \int_0^\infty e^{-(s+\lambda-\lambda z+\alpha)x}$ dB(x) is the Laplace – Steiljes transform of service time.

$$\overline{V}(z,s) = \overline{V}(0,z,s)\left[\frac{1-\overline{V}(S+\lambda-\lambda z)}{s+\lambda-\lambda z}\right] \quad (4.37)$$

Where $\overline{V}(s+\lambda-\lambda z)=\int_0^{\infty} e^{-(s+\lambda-\lambda z)x}$ dV(x) is the Laplace – Steiltjes transform of the vacation time.

$$\overline{R}^{(1)}(z,s)=\overline{R}^{(1)}(0,z,s)\left[\frac{1-\overline{R_1}(s+\lambda-\lambda z)}{s+\lambda-\lambda z}\right] \quad (4.38)$$

Where $\overline{R_1}(s+\lambda-\lambda z)=\int_0^{\infty} e^{-(s+\lambda-\lambda z)x}$ $dR_1(x)$ is the Laplace – Steiltjes transform of the phase 1 repair time.

$$\overline{R}^{(2)}(z,s)=\overline{R}^{(2)}(0,z,s)\left[\frac{1-\overline{R_2}(s+\lambda-\lambda z)}{s+\lambda-\lambda z}\right] \quad (4.39)$$

Where $\overline{R_2}(s+\lambda-\lambda z)=\int_0^{\infty} e^{-(s+\lambda-\lambda z)x}$ $dR_2(x)$ is the Laplace – Steiltjes transform of the phase 2 repair time.

$$\overline{R}^{(3)}(z,s)=\overline{R}^{(3)}(0,z,s)\left[\frac{1-\overline{R_3}(s+\lambda-\lambda z)}{s+\lambda-\lambda z}\right] \quad (4.40)$$

Where $\overline{R_3}(s+\lambda-\lambda z)=\int_0^{\infty} e^{-(s+\lambda-\lambda z)x}$ $dR_3(x)$ is the Laplace – Steiltjes transform of the phase 3 repair time.

Now we shall determine the following integrals $\int_0^{\infty}\overline{P}(x,z,s)\mu(x)dx$

$$\int_0^{\infty}\overline{V}(x,z,s)\gamma(x)dx,\ \int_0^{\infty}\overline{R}^{(1)}(x,z,s)\beta_1(x)dx,\ \int_0^{\infty}\overline{R}^{(2)}(x,z,s)\beta_2(x)dx$$

and $\int_0^{\infty}\overline{R}^{(3)}(x,z,s)\beta_3(x)dx$.

For this purpose we multiply equations (4.31) – (4.35) by $\mu(x),\gamma(x),\beta_1(x)$, $\beta_2(x)$ and $\beta_3(x)$, integrate with respect to x and obtain

$$\int_0^{\infty}\overline{P}(x,z,s)\mu(x)dx=\overline{P}(0,z,s)\overline{B}(s+\lambda-\lambda z+\alpha), \quad (4.41)$$

$$\int_0^{\infty}\overline{V}(x,z,s)\gamma(x)dx=\overline{V}(0,z,s)\overline{V}(s+\lambda-\lambda z), \quad (4.42)$$

$$\int_0^{\infty}\overline{R}^{(1)}(x,z,s)\beta_1(x)dx=\overline{R}^{(1)}(0,z,s)\overline{R_1}(s+\lambda-\lambda z), \quad (4.43)$$

$$\int_0^{\infty}\overline{R}^{(2)}(x,z,s)\beta_2(x)dx=\overline{R}^{(2)}(0,z,s)\overline{R_2}(s+\lambda-\lambda z), \quad (4.44)$$

$$\int_0^{\infty}\overline{R}^{(3)}(x,z,s)\beta_3(x)dx=\overline{R}^{(3)}(0,z,s)\overline{R_3}(s+\lambda-\lambda z). \quad (4.45)$$

Now using equations (4.41) – (4.45) into equations (4.26) – (4.30), we get on simplifying

$$z\overline{P}(0,z,s)=(1-s\overline{Q}(s))+\lambda(z-1)\overline{Q}(s)+\overline{R}^{(3)}(0,z,s)\overline{R_3}(s+\lambda-\lambda z)$$
$$+\overline{V}(0,z,s)\overline{V}(s+\lambda-\lambda z), \quad (4.46)$$

$$\overline{V}(0,z,s)=\overline{P}(0,z,s)\overline{B}(s+\lambda-\lambda z+\alpha), \quad (4.47)$$

$$\overline{R}^{(1)}(0,z,s)=\alpha z\int_0^{\infty}\overline{P}(x,z,s)dx=\alpha z\overline{P}(z,s), \quad (4.48)$$

$$\overline{R}^{(2)}(0,z,s)=\alpha z\overline{P}(z,s)\overline{R_1}(s+\lambda-\lambda z), \quad (4.49)$$

$$\overline{R}^{(3)}(0,z,s)=\alpha z\overline{P}(z,s)\overline{R_1}(s+\lambda-\lambda z)\overline{R_2}(s+\lambda-\lambda z). \quad (4.50)$$

Now using equations (4.47) and (4.50) in equation (4.46) and then simplifying we get

$$\overline{P}(0,z,s)=\left[\frac{(1-s\overline{Q}(s))+\lambda(z-1)\overline{Q}(s)}{\overline{D}(z,s)}\right], \quad (4.51)$$

where

$$\overline{D}(z,s) = z - \overline{B}(s+\lambda-\lambda z+\alpha)\overline{V}(s+\lambda-\lambda z) - \alpha z\overline{R_1}(s+\lambda-\lambda z)$$
$$\overline{R_2}(s+\lambda-\lambda z)\,\overline{R_3}(s+\lambda-\lambda z)[1-\overline{B}(s+\lambda-\lambda z+\alpha)]. \quad (4.52)$$

Using equations (4.52) and (4.53) in equation (4.37) we get

$$\overline{P}(z,s) = \left[\frac{\overline{N_1}(z,s)}{\overline{D_1}(z,s)}\right], \quad (4.53)$$

where

$$\overline{N_1}(z,s) = (1-s\overline{Q}(s)) + \lambda(z-1)\overline{Q}(s)[1-\overline{B}(s+\lambda-\lambda z+\alpha)], \quad (4.54)$$

$$\overline{D}_1(z,s) = (S+\lambda-\lambda z+\alpha)[z-\overline{B}(s+\lambda-\lambda z+\alpha)\overline{V}(s+\lambda-\lambda z)]$$
$$-\alpha z\overline{R_1}(s+\lambda-\lambda z)\overline{R_2}(s+\lambda-\lambda z)\overline{R_3}(s+\lambda-\lambda z)\left[\frac{1-\overline{B}(s+\lambda-\lambda z+\alpha)}{(s+\lambda-\lambda z+\alpha)}\right]. \quad (4.55)$$

Using equations (4.52) to (4.55) into equations (4.37) – (4.40), we get

$$\overline{V}(z,s) = \overline{P}(0,z,s)\overline{B}(s+\lambda-\lambda z+\alpha)\left[\frac{1-\overline{V}(S+\lambda-\lambda z)}{s+\lambda-\lambda z}\right], \quad (4.56)$$

where $\overline{P}(0,z,s)$ are given by equations (4.51) and (4.52).

$$\overline{R}^{(1)}(z,s) = \alpha z\overline{P}(z,s)\left[\frac{1-\overline{R_1}(S+\lambda-\lambda z)}{s+\lambda-\lambda z}\right]. \quad (4.57)$$

$$\overline{R}^{(2)}(z,s) = \alpha z\overline{P}(z,s)\overline{R_1}(S+\lambda-\lambda z)\left[\frac{1-\overline{R_2}(S+\lambda-\lambda z)}{s+\lambda-\lambda z}\right], \quad (4.58)$$

$$\overline{R}^{(3)}(z,s) = \alpha z\overline{P}(z,s)\overline{R_1}(S+\lambda-\lambda z)\overline{R_2}(S+\lambda-\lambda z)$$
$$\left[\frac{1-\overline{R_3}(S+\lambda-\lambda z)}{s+\lambda-\lambda z}\right]. \quad (4.59)$$

5. STEADY STATE SOLUTION

In this section, we shall derive the steady state probability distribution for our queueing model. To define the steady state probabilities, we suppress the argument t wherever it appears in the time-dependent analysis. This can be obtained by applying the well-known Tauberian property,

$$\lim_{n\to 0} s\overline{f}(s) = \lim_{t\to\infty} f(t) \quad (5.1)$$

In order to determine $\overline{P}(z,s)$, $\overline{V}(z,s)$, $\overline{R}^{(1)}(z,s)$, $\overline{R}^{(2)}(z,s)$ and $\overline{R}^{(3)}(z,s)$ completely, we have yet to determine the unknown Q. For that purpose, we shall use the normalizing condition

$$P(1) + V(1) + R^{(1)}(1) + R^{(2)}(1) + R^{(3)}(1) + Q = 1\ . \quad (5.2)$$

Thus multiplying both sides of equations (4.53), (4.56), (4.57), (4.58) and (4.59) by s, taking limit as $s \to 0$, applying property (5.1) and simplifying we have

$$P(z) = \frac{\lambda(z-1)Q[1-\overline{B}(\lambda-\lambda z+\alpha)]}{D(z)}, \quad (5.3)$$

$$V(z) = \frac{\lambda(z-1)Q(\lambda-\lambda z+\alpha)}{D(z)}, \quad (5.4)$$

$$R^{(1)}(z) = \frac{\alpha Q[1-\overline{B}(\lambda-\lambda z+\alpha)][\overline{R_1}(\lambda-\lambda z)-1]}{D(z)}, \quad (5.5)$$

$$R^{(2)}(z) = \frac{\alpha Q[1-\overline{B}(\lambda-\lambda z+\alpha)]\overline{R_1}(\lambda-\lambda z)[\overline{R_2}(\lambda-\lambda z)-1]}{D(z)}, \quad (5.6)$$

$$R^{(3)}(z) = \frac{\alpha Q[1-\overline{B}(\lambda-\lambda z+\alpha)]\overline{R_1}(\lambda-\lambda z)\overline{R_2}(\lambda-\lambda z)[\overline{R_3}(\lambda-\lambda z)-1]}{D(z)}. \quad (5.7)$$

where

$$D(z) = (\lambda - \lambda z + \alpha)\left[z - \overline{B}(\lambda - \lambda z + \alpha)\overline{V}(\lambda - \lambda z)\right] - \alpha z \overline{R_1}(\lambda - \lambda z)\overline{R_2}(\lambda - \lambda z)$$
$$\overline{R_3}(\lambda - \lambda z)\left[1 - \overline{B}(\lambda - \lambda z + \alpha)\right]. \tag{5.8}$$

We see that for $z = 1$, $P(z), V(z), R^{(1)}(z), R^{(2)}(z)$ and $R^{(3)}(z)$ in equations (5.3) - (5.8) are indeterminate of the $\frac{0}{0}$ form. Therefore, we apply L'Hopital's rule on equations (5.3) - (5.8) using the fact that

$$\overline{B}(0) = 1, -\overline{B}'(0) = \frac{1}{\mu}, \ \overline{V}(0) = 1, -\overline{V}'(0) = \frac{1}{\gamma} = E(v),$$
$$\overline{R_1}(0) = 1, -\overline{R_1}'(0) = \frac{1}{\beta_1} = E(r_1), \overline{R_2}(0) = 1, -\overline{R_2}'(0) = \frac{1}{\beta_2} = E(r_2) \text{ and}$$
$$\overline{R_3}(0) = 1, -\overline{R_3}'(0) = \frac{1}{\beta_3} = E(r_3).$$

Thus on simplifying we have

$$P(1) = \frac{[1-\overline{B}(\alpha)]\lambda Q}{D(1)}, \tag{5.9}$$

$$V(1) = \frac{\alpha\lambda Q\overline{B}(\alpha)E(v)}{D(1)}. \tag{5.10}$$

$$R^{(1)}(1) = \frac{\alpha\lambda Q[1-\overline{B}(\alpha)]E(r_1)}{D(1)}, \tag{5.11}$$

$$R^{(2)}(1) = \frac{\alpha\lambda Q[1-\overline{B}(\alpha)]E(r_2)}{D(1)}, \tag{5.12}$$

$$R^{(3)}(1) = \frac{\alpha\lambda Q[1-\overline{B}(\alpha)]E(r_3)}{D(1)}. \tag{5.13}$$

Where

$$D(1) = \alpha\overline{B}(\alpha) - \lambda\left[1 - \overline{B}(\alpha)\right] + \alpha\lambda\overline{B}(\alpha)\overline{V}'(0) + \alpha\lambda\overline{B}(\alpha)\overline{R_1}'(0)$$
$$+\alpha\lambda\overline{B}(\alpha)\overline{R_2}'(0) + \alpha\lambda\overline{B}(\alpha)\overline{R_3}'(0). \tag{5.14}$$

Now since we must have $P(1) + V(1) + R^{(1)}(1) + R^{(2)}(1) + R^{(3)}(1) + Q = 1$, we have

$$Q = 1 - \frac{\lambda}{\alpha\overline{B}(\alpha)}\left\{\begin{matrix}\left[1 - \overline{B}(\alpha)\right] - \alpha\overline{B}(\alpha)\overline{V}'(0) - \alpha\left[1 - \overline{B}(\alpha)\right]\overline{R_1}'(0) \\ -\alpha\left[1 - \overline{B}(\alpha)\right]\overline{R_2}'(0) - \alpha\left[1 - \overline{B}(\alpha)\right]\overline{R_3}'(0)\end{matrix}\right\}. \tag{5.15}$$

Equation (5.15) gives the steady state probability that there is no customer in the system and the server is idle. Also from equation (5.15), we obtain ρ, the utilization factor of the system as where

$$\rho = 1 - Q$$
$$= \frac{\lambda}{\alpha\overline{B}(\alpha)}\left\{\begin{matrix}\left[1 - \overline{B}(\alpha)\right] - \alpha\overline{B}(\alpha)\overline{V}'(0) - \alpha\left[1 - \overline{B}(\alpha)\right]\overline{R_1}'(0) \\ -\alpha\left[1 - \overline{B}(\alpha)\right]\overline{R_2}'(0) - \alpha\left[1 - \overline{B}(\alpha)\right]\overline{R_3}'(0)\end{matrix}\right\} < 1. \tag{5.16}$$

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