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IMPLEMENTATION OF KALMAN FILTER ALGORITHM ON REDUCED MODELS WITH LINEAR MATRIX INEQUALITY METHOD AND ITS APPLICATION TO HEAT CONDUCTION PROBLEMS

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ABSTRACT

In this paper, we discuss the model reduction and estimation of state variables of the heat conduction system by using Linear Matrix Inequality method and Kalman filter algorithm. We aim to obtain accurate estimation with short computing time. First, we construct a reduced model by using Linear Matrix Inequality method. Further, we apply state variables estimation steps of discrete stochastic dynamical systems by using Kalman filter algorithm on the reduced model.

Keywords: Estimation, Kalman Filter, Model Reduction, Linear Matrix Inequality.

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1. INTRODUCTION

Estimation of state variable in a system is quite important. One of the estimation algorithm is Kalman filter. Kalman Filter (KF) is an algorithm that combines models and measurements. The latest measurement data is an important part of the KF algorithm because the latest data will correct the prediction results, so the estimation results are always close to the actual conditions [14]. Kalman filter was applied in many problems, such as estimation of river water levels [1], estimation of some environmental problems [2], estimation of heat distribution [3], and many others.

In generally, estimation method aims to obtain accurate result with short computing time. The computing time is influenced by order of the system. Therefore to reduce computational time, it can be done by replacing a high-order system with a simple system with smaller orders without significant errors. Models with smaller orders are called reduced model. The way to get a reduced model is called model reduction [9]. Model reduction methods have been widely developed, including the Balanced Truncation (BT) method, the Singular Pertubation Approximation (SPA) method [7, 8, 17], and the Linear Matrix Inequality (LMI) method [4, 5, 9, 11]. The LMI method produces an error reduction, that is measured by the $\mathcal{H}_\infty$ norm, much smaller than the upper bounds of the error of model reduction [6].

Research that combines model reduction methods with estimation methods to obtain accurate estimation results and short computation time has also been carried out, such as combining KF algorithm and BT methods [3, 12, 15], also combining KF algorithm and SPA methods [16]. In this paper, we combine KF algorithm and LMI methods and then we use it to estimate state variables of the heat conduction system.

2. PRELIMINARIES

2.1. Model Reduction with Linear Matrix Inequality Method

A discrete linear time invariant system is given by

$$x_{k+1} = \mathbf{A}x_k + \mathbf{B}u_k\,, \tag{1}$$

$$y_k = \mathbf{C}x_k + \mathbf{D}u_k\,. \tag{2}$$

$$k = 0, 1,2,3,4, \ldots$$

where $x_k \in \mathbb{R}^n$ is the state variables, $u_k \in \mathbb{R}^m$ is the input variables, $y_k \in \mathbb{R}^p$ is the output variables and $\mathbf{A} \in \mathbb{R}^{n\times n}, \mathbf{B} \in \mathbb{R}^{n\times m}, \mathbf{C} \in \mathbb{R}^{p\times n}$, and $\mathbf{D} \in \mathbb{R}^{p\times m}$ are constant matrices. In this paper, such system (1) and (2) is written as $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ for simplicity.

In order to directly relate the input and output variables, we can use the transfer function of the system that can be obtained by using the following formula :

$$G(z) = \mathbf{C}(zI - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D},$$

with the realization of state space as follows:

$$G(z) = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right].$$

On the other hand, the reduced space state of the discrete linear time invariant system equation having the order r with $r < n$ is given by

$$x_{r_{k+1}} = \mathbf{A}_r x_{r_k} + \mathbf{B}_r u_{r_k}, \tag{3}$$

$$y_{r_k} = \mathbf{C}_r x_{r_k} + \mathbf{D}_r u_{r_k}\,, \tag{4}$$

where $x_k \in \mathbb{R}^r$ is the state variables, $u_k \in \mathbb{R}^m$ is the input variables, $y_k \in \mathbb{R}^p$ is the output variables and $\mathbf{A} \in \mathbb{R}^{r\times r}, \mathbf{B} \in \mathbb{R}^{r\times m}, \mathbf{C} \in \mathbb{R}^{p\times r}$, and $\mathbf{D} \in \mathbb{R}^{p\times m}$ are constant matrices. Furthermore systems (3) and (4) is written as $(\mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r, \mathbf{D}_r)$.

The transfer function of the reduced system $(\mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r, \mathbf{D}_r)$ can be obtained by using the following formula :

$$G_r(z) = \mathbf{C}_r(zI - \mathbf{A}_r)^{-1}\mathbf{B}_r + \mathbf{D}_r,$$

with the form of realization of state space as follows:

$$G_r(z) = \left[\begin{array}{c|c} \mathbf{A}_r & \mathbf{B}_r \\ \hline \mathbf{C}_r & \mathbf{D}_r \end{array}\right].$$

Furthermore, the error caused by the reduction model is denoted $E(z)$ and is defined as

$$E(z) = G(z) - G_r(z) = \left[\begin{array}{c|c} \mathbf{A}_e & \mathbf{B}_e \\ \hline \mathbf{C}_e & \mathbf{D}_e \end{array}\right].$$

The transfer function of $E(z)$ can be obtained by using the following formula: $G_e(z) = \mathbf{C}_e(zI - \mathbf{A}_e)^{-1}\mathbf{B}_e + \mathbf{D}_e$,

With the form of realization of state space as follows:

$$E(z) = \left[\begin{array}{cc|c} \mathbf{A} & 0 & \mathbf{B} \\ 0 & \mathbf{A}_r & \mathbf{B}_r \\ \hline \mathbf{C} & -\mathbf{C}_r & \mathbf{D} - \mathbf{D}_r \end{array}\right].$$

Based on the realization of the state of space for $E(z)$ can be obtained

$$\mathbf{A}_e = \begin{bmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{A}_r \end{bmatrix}, \mathbf{B}_e = \begin{bmatrix} \mathbf{B} \\ \mathbf{B}_r \end{bmatrix}, \mathbf{C}_e = [\mathbf{C} \quad -\mathbf{C}_r], \text{dan } \mathbf{D}_e = [\mathbf{D} - \mathbf{D}_r]$$

Model reduction using LMI method is performed on linear matrix inequality so that the form $\|E(z)\|_\infty$ must be transformed into the form of a linear matrix inequality by using Bounded Real Lemma.

The statement on Bounded Real Lemma is as follows.

Let nonnegative scalar $\gamma > 0$ and $E(z) = \left[\begin{array}{c|c} \mathbf{A}_e & \mathbf{B}_e \\ \hline \mathbf{C}_e & \mathbf{D}_e \end{array}\right]$, then $\|E(z)\|_\infty < \gamma$ if and only if there are $P > 0$ such that

$$\begin{bmatrix} \mathbf{A}_e & \mathbf{B}_e \\ \mathbf{C}_e & \mathbf{D}_e \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathbf{A}_e & \mathbf{B}_e \\ \mathbf{C}_e & \mathbf{D}_e \end{bmatrix} < \begin{bmatrix} P & 0 \\ 0 & \gamma^2 I \end{bmatrix}. \quad (5)$$

The P matrix is partitioned to correspond to the matrix $\mathbf{A}_e, \mathbf{B}_e, \mathbf{C}_e$, and $\mathbf{D}_e$.

So that the size of the matrix P corresponds to the matrix $\mathbf{A}_e, \mathbf{B}_e, \mathbf{C}_e$, and $\mathbf{D}_e$, then the P matrix can be partitioned into

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} > 0,$$

with P_{11}, P_{12} and P_{22} each is a real matrix measuring $n \times n$, $n \times r$, and $r \times r$ so P is a positive definite matrix of size $(n + r) \times (n + r)$ [5].

The following Theorem 1 states the necessary and sufficient conditions for the existence of reduction model with the LMI method.

Theorem 1.[5]. Given the system $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ with the transfer function $G(z) \in \mathcal{RH}_\infty$ and the realization of the space of the system is minimal, i.e.

$$G(z) = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right],$$

then there is an reduced system $(\mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r, \mathbf{D}_r)$ with the transfer function $G_r(z) \in \mathcal{RH}_\infty$ that satisfy $\|G(z) - G_r(z)\|_\infty < \gamma$ if and only if there is $X_{11} \in \mathcal{S}_n$, $P_{11} \in \mathcal{S}_n$, $P_{12} \in \mathbb{R}^{n\times r}$, and $P_{22} \in \mathcal{S}_r$ that satisfy the following matrix inequalities :

$$-X_{11} + \mathbf{A}X_{11}\mathbf{A}^T + \frac{1}{\gamma^2}\mathbf{B}\mathbf{B}^T < 0 \tag{6a}$$

$$-P_{11} + \mathbf{A}^T P_{11}\mathbf{A} + \mathbf{C}^T\mathbf{C} < 0 \tag{6b}$$

with $$X_{11} = (P_{11} - P_{12}P_{22}^{-1}P_{12}^T)^{-1} \tag{6c}$$

and $\mathcal{S}_n$ denotes the set of positive definite matrices of size $n \times n$.

Definition 2. (Controllability and Observability Gramians). The controllability gramian of system $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is $M \coloneqq \sum_{k=0}^{\infty} \mathbf{A}^k \mathbf{B}\mathbf{B}^T(\mathbf{A}^T)^k$. The observability gramian of system $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is $N \coloneqq \sum_{k=0}^{\infty}(\mathbf{A}^T)^k \mathbf{C}^T\mathbf{C}\mathbf{A}^k$.

Controllability and observability gramians are a positive definite and a single solution of the following Lyapunov equation :

$$\mathbf{A}M\mathbf{A}^T + \mathbf{B}\mathbf{B}^T - M = 0,$$

and

$$\mathbf{A}^T N\mathbf{A} + \mathbf{C}^T\mathbf{C} - N = 0.$$

Definition 3. The Hankel singular value of the system $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ with the transfer function $G(z)$ is defined as

$$\sigma_i = \sqrt{\lambda_i(MN)}$$

with $\lambda_i(MN)$ declaring the largest eigenvalue of the matrix MN for $i = 1, 2, \ldots, n$.

In the following, we assume that the realization $G(z) = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]$ is already balanced, i.e., its controllability and observability Gramians are equal and diagonal. Hence by denoting the balanced Gramian by Σ, the state space matrices **A**, **B** and **C** satisfy :

$$\mathbf{A}\Sigma\,\mathbf{A}^T + \mathbf{B}\mathbf{B}^T - \Sigma = 0,$$

and

$$\mathbf{A}^T\Sigma\,\mathbf{A} + \mathbf{C}^T\mathbf{C} - \Sigma = 0,$$

with $\Sigma = \mathrm{diag}(\sigma_1 I_{k_1}, \ldots, \sigma_l I_{k_l}, \sigma_{l+1} I_{k_{l+1}}, \ldots, \sigma_m I_{k_m})$, $\sigma_1 > \cdots > \sigma_l > \sigma_{l+1} > \cdots > \sigma_m > 0$.

Note that k_i is the multiplicity of σ_i and $k_1 + \cdots + k_l + k_{l+1} + \cdots + k_m = n$.

As is well-known, the diagonal entries of Σ are called the Hankel singular values of the system $G(z)$ and plays a key role in the balanced truncation method.

Based on Theorem 1, a lower limit of the error of the model order reduction results can be derived. The lower limit of $\|G(z) - G_r(z)\|_\infty$ is stated in the following Theorem 4.

Theorem 4. [5]. Given the system $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ with the transfer function $G(z) = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right] \in \mathcal{RH}_\infty$ which has the singular value Hankel $\sigma_1 \geq \cdots \geq \sigma_r \geq \sigma_{r+1} \geq \cdots \geq \sigma_n > 0$. Then, for all reduced system $(\mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r, \mathbf{D}_r)$ which has $r < n$ with the transfer function $G_r(z) = \left[\begin{array}{c|c} \mathbf{A}_r & \mathbf{B}_r \\ \hline \mathbf{C}_r & \mathbf{D}_r \end{array}\right] \in \mathcal{RH}_\infty$ we have

$$\|G(z) - G_r(z)\|_\infty \geq \sigma_{r+1}. \tag{7}$$

Furthermore, the infimum of $\left\|G(z) - G_{n-k_m}(z)\right\|_\infty$ is stated in the following theorem.

Theorem 5. [5]. Given a system $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ with transfer function $G(z) \in \mathcal{RH}_\infty$ which has the Hankel singular value $\sigma_1 > \cdots > \sigma_l > \sigma_{l+1} > \cdots > \sigma_m > 0$, with σ_1 as many as k_1,

σ_2 as many as k_2, and so on until σ_m as many as k_m so that $k_1 + k_2 + \cdots + k_m = n$. Then, for arbitrary $\gamma > \sigma_m$, there exists an reduced system $\left(\mathbf{A}_{n-k_m}, \mathbf{B}_{n-k_m}, \mathbf{C}_{n-k_m}, \mathbf{D}_{n-k_m}\right)$ which has the order $n - k_m$ with the transfer function $G_{n-k_m}(z) \in \mathcal{RH}_\infty$ that satisfies

$$\left\|G(z) - G_{n-k_m}(z)\right\|_\infty < \gamma. \tag{8}$$

One important implication of Theorem 5 is that, in the case where $r = n - k_m$, we can fix the matrix variable P_{11} and P_{22} to be constant, with

$$P_{12} = \begin{bmatrix} I_{n-k_m} \\ 0_{k_m \times (n-k_m)} \end{bmatrix}, P_{22} = \operatorname{diag}\left((\sigma_1 - \tfrac{\sigma_m^2}{\sigma_1})^{-1} I_{k_1}, \ldots, (\sigma_{m-1} - \tfrac{\sigma_m^2}{\sigma_{m-1}})^{-1} I_{k_{m-1}}\right) > 0 \tag{9}$$

The reduced system $(\mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r, \mathbf{D}_r)$, $r = n - k_m$ which has the order $n - k_m$ with the transfer function $G_{n-k_m}(z)$ that minimizes $\left\|G(z) - G_{n-k_m}(z)\right\|_\infty$ can be obtained by the following theorem.

Theorem 6. [5]. The reduced system $(\mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r, \mathbf{D}_r)$, $r = n - k_m$ which has the order $n - k_m$ with the transfer function $G_{n-k_m}(z)$ that minimizes $\left\|G(z) - G_{n-k_m}(z)\right\|_\infty$ can be obtained by two-step procedure :

Minimize γ^2 subject to the LMIs :

$$\begin{bmatrix} P_{11} & P_{12} Q_{22} \\ Q_{22} P_{12}^T & Q_{22} \end{bmatrix} > 0 ,$$

$$\begin{bmatrix} -(P_{11} - P_{12} Q_{22} P_{12}^T) & (P_{11} - P_{12} Q_{22} P_{12}^T)\mathbf{A} & (P_{11} - P_{12} Q_{22} P_{12}^T)\mathbf{B} \\ \mathbf{A}^T (P_{11} - P_{12} Q_{22} P_{12}^T)^T & -(P_{11} - P_{12} Q_{22} P_{12}^T) & 0 \\ \mathbf{B}^T (P_{11} - P_{12} Q_{22} P_{12}^T)^T & 0 & -\gamma^2 I \end{bmatrix} < 0 , \tag{10}$$

$$-P_{11} + \mathbf{A}^T P_{11} \mathbf{A} + \mathbf{C}^T \mathbf{C} < 0,$$

where $P_{11} \in \mathcal{S}_n$ and $Q_{22} \in \mathcal{S}_{n-k_m}$ are matrix variables to be determined whereas $P_{12} \in \mathbb{R}^{n \times (n-k_m)}$ is a constant matrix given by $P_{12} = \begin{bmatrix} I_{n-k_m} \\ 0_{k_m,(n-k_m)} \end{bmatrix}$. For the subsequent step, define $\tilde{P} = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & Q_{22}^{-1} \end{bmatrix}$ and denote the optimal value of γ by γ_{opt}.

Obtained $(\mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r, \mathbf{D}_r)$ by solving (5), where P is fixed to $\tilde{P}$ and γ to γ_{opt}. The LMIs in (10) given in the first step can be obtained from (6) by defining $Q_{22} := P_{22}^{-1}$ and applying Schur complements arguments. The coefficient matrices $(\mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r, \mathbf{D}_r)$ in the second step can be constructed also from $\tilde{P}$ by analytic formulas given in [9, 10, 11]. It should be noted that, since the choice of P_{12} depends on the state space realizations, the result in Theorem 6 is valid only if $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is balanced. From the Schur complement arguments, we can rewrite (5) equivalently as follows :

$$\begin{bmatrix} -P_{11} & -P_{12} & 0 & \mathbf{A}^T P_{11} & \mathbf{A}^T P_{12} & \mathbf{C}^T \\ * & -P_{22} & 0 & \mathbf{A}_r^T P_{12}^T & \mathbf{A}_r^T P_{22} & -\mathbf{C}_r^T \\ * & * & -\gamma^2 I_p & \mathbf{B}^T P_{11} + \mathbf{B}_r^T P_{12}^T & \mathbf{B}^T P_{12} + \mathbf{B}_r^T P_{22} & \mathbf{D}^T - \mathbf{D}_r^T \\ * & * & * & -P_{11} & -P_{12} & 0 \\ * & * & * & * & -P_{22} & 0 \\ * & * & * & * & * & -I_q \end{bmatrix} < 0. \tag{11}$$

By the similarity transformation $\bar{\mathbf{A}}_r := P_{22} \mathbf{A}_r P_{22}^{-1}$, $\bar{\mathbf{B}}_r := P_{22} \mathbf{B}_r$, and $\bar{\mathbf{C}}_r := \mathbf{C}_r P_{22}^{-1}$, we see that there exist $(\mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r, \mathbf{D}_r)$ that satisfy (11) if and only if

$$\begin{bmatrix} -P_{11} & -P_{12} & 0 & \mathbf{A}^T P_{11} & \mathbf{A}^T P_{12} & \mathbf{C}^T \\ * & -P_{22} & 0 & P_{22}\,\overline{\mathbf{A}}_r^T P_{22}^{-1} P_{12}^T & P_{22}\,\overline{\mathbf{A}}_r^T & -P_{22}\,\overline{\mathbf{C}}_r^T \\ * & * & -\gamma^2 I_p & \mathbf{B}^T P_{11} + \overline{\mathbf{B}}_r^T P_{22}^{-1} P_{12}^T & \mathbf{B}^T P_{12} + \overline{\mathbf{B}}_r^T & \mathbf{D}^T - \overline{\mathbf{D}}_r^T \\ * & * & * & -P_{11} & -P_{12} & 0 \\ * & * & * & * & -P_{22} & 0 \\ * & * & * & * & * & -I_q \end{bmatrix} < 0. \quad (12)$$

By the congruence transformation with $\mathrm{diag}(I, Q_{22}, I, I, Q_{22}, I)$ where $Q_{22} = P_{22}^{-1}$, the above inequality reduces to

$$\begin{bmatrix} -P_{11} & -P_{12}Q_{22} & 0 & \mathbf{A}^T P_{11} & \mathbf{A}^T P_{12} Q_{22} & \mathbf{C}^T \\ * & -Q_{22} & 0 & \overline{\mathbf{A}}_r^T Q_{22} P_{12}^T & \overline{\mathbf{A}}_r^T Q_{22} & -\overline{\mathbf{C}}_r^T \\ * & * & -\gamma^2 I_p & \mathbf{B}^T P_{11} + \overline{\mathbf{B}}_r^T Q_{22} P_{12}^T & \mathbf{B}^T P_{12} Q_{22} + \overline{\mathbf{B}}_r^T Q_{22} & \mathbf{D}^T - \overline{\mathbf{D}}_r^T \\ * & * & * & -P_{11} & -P_{12} Q_{22} & 0 \\ * & * & * & * & -Q_{22} & 0 \\ * & * & * & * & * & -I_q \end{bmatrix} < 0. \quad (13)$$

Here, since we can fix the matrix variable P_{12} to be constant as in (9), we see that the above inequality is an LMI with respect to the matrix variables P_{11}, Q_{22}, and $\widetilde{\mathbf{A}}_r := Q_{22}\,\overline{\mathbf{A}}_r$, $\widetilde{\mathbf{B}}_r := Q_{22}\,\overline{\mathbf{B}}_r$, $\overline{\mathbf{B}}_r, \mathbf{D}_r$. Once these variables have been found, the optimal reduced order model can be constructed by

$$G_r(z) = \begin{bmatrix} Q_{22}^{-1}\widetilde{\mathbf{A}}_r & Q_{22}^{-1}\widetilde{\mathbf{B}}_r \\ \overline{\mathbf{C}}_r & \mathbf{D}_r \end{bmatrix}. \quad (14)$$

2.2. Kalman Filter Algorithm for Discrete Systems

Kalman filter is one method to estimate state variable from stochastic dynamic system first introduced by Rudolf E. Kalman in 1960. Estimation using this method is done by predicting state variable in dynamic systems which are then corrected using measurement data [14]. In system modeling there is no mathematical model of a perfect system, because there are noise factors in each system. Therefore, it is necessary to add stochastic factors to the deterministic system $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ in equations (1) and (2) in the form of noise system and noise measurement, in order to obtain the following stochastic dynamic system:

$$x_{k+1} = \mathbf{A}x_k + \mathbf{B}u_k + Gw_k, \quad (15)$$

$$z_k = \mathbf{C}x_k + \mathbf{D}u_k + v_k, \quad (16)$$

with w_k and v_k are noise system and noise measurement, and each is a stochastic scale. Noise system and noise measurement are assumed to be normally distributed with zero mean and covariance, respectively Q_k and R_k.

The KF algorithm for discrete stochastic dynamic systems [13] can be written as follows:

Given

$$x_{k+1} = \mathbf{A}x_k + \mathbf{B}u_k + Gw_k, \quad (17)$$

$$z_k = \mathbf{C}x_k + \mathbf{D}u_k + v_k \quad (18)$$

$$x_0 \sim N(\bar{x}_0, P_{x_0}); \;\; w_k \sim N(0, Q); \;\; v_k \sim N(0, R).$$

Initialization

$$P_0 = P_{x_0}; \;\; \hat{x}_0 = \bar{x}_0. \quad (19)$$

Prediction Stage (*time update*)

$$\text{Error Covariance} : P_{k+1}^{-} = \mathbf{A}P_k\mathbf{A}^T + GQG^T. \quad (20)$$

$$\text{Estimation} \quad : \hat{x}_{k+1}^{-} = \mathbf{A}\hat{x}_k + \mathbf{B}u_k. \quad (21)$$

Correction Stage (*measurement update*)

$$\text{Error Covariance} : P_{k+1} = [(P_{k+1}^{-})^{-1} + \mathbf{C}^T R^{-1}\mathbf{C}]^{-1} \quad (22)$$

$$\text{Estimastion} : \hat{x}_{k+1} = \hat{x}_{k+1}^{-} + P_{k+1}H^T R^{-1}(z_{k+1} - \mathbf{C}\hat{x}_{k+1}^{-}) \quad (23)$$

$$\text{Kalman gain } K_{k+1} = P_{k+1}^{-}\mathbf{C}^T(\mathbf{C}P_{k+1}^{-}\mathbf{C}^T + R)^{-1}, \quad (24)$$

$$\text{Error Covariance} : P_{k+1} = (I - K_{k+1}\mathbf{C})P_{k+1}^{-}. \quad (25)$$

$$\text{Estimation} : \hat{x}_{k+1} = \hat{x}_{k+1}^{-} + K_{k+1}(z_{k+1} - \mathbf{C}\hat{x}_{k+1}^{-}). \quad (26)$$

Return to the Prediction Stage.

3. NUMERICAL COMPUTATION

3.1. Mathematical Model for Heat Conduction Systems

Given the problem of heat conduction in a straight wire with length L and heat conduction coefficient α. The x axis is chosen to express the longitudinal direction of the wire rod with $x = 0$ and $x = L$ denoting the position of the ends of the wire rod.

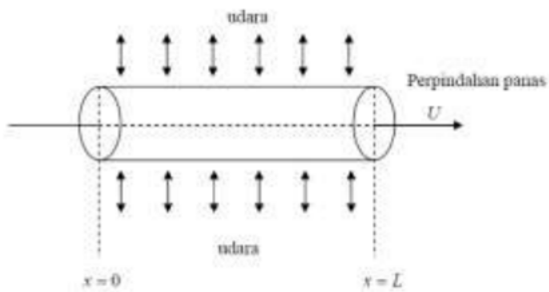


Figure 1 Heat conduction in a wire rod [3].

It is assumed that the sides of the wire rod are completely insulated, meaning that no heat can penetrate the sides of the wire rod. The heat flowing on the wire rod is only influenced by position and time. The temperature or heat on the wire rod is denoted by u, the position along the wire rod is denoted by x, and the time is denoted by t. The temperature variation in the wire rod is expressed in the following heat conduction equation :

$$\frac{\partial U}{\partial t} = \alpha\frac{\partial^2 U}{\partial x^2}, (0 < x < L, t > 0) \quad (27)$$

with α is the thermal diffusity of the wire material [3]. Furthermore, it is assumed that at the right end of the wire rod is perfectly isolated, meaning that there is no heat change at the position $x = L$.

Equation (27) is applied to a metal rod heated at the base of the stem ($x = 0$), while the rod end ($x = L$) is fixed. The heat at the base of the stem will propagate to the end of the stem. Using equation (16) can be estimated the temperature along the metal (x) and at any time (t).

In Figure 1, a heat conducting rod has an initial temperature distribution at $t = 0$, and at its ends has a temperature which is a function of time. The temperature distribution $U(x, t)$ in the stem at time $t > 0$ can be calculated assuming that the physical properties of the rod are constant. Problems can be presented in the form of differential equations with initial conditions and boundaries.

Equation (16) applies to regions $0 < x < L$ and $0 < t < \tau$, where τ is the total count time, while the initial conditions and boundaries are:

$$\begin{aligned} U(x,0) &= f(x) && ;0 \le x \le L \\ U(0,t) &= g_0(t) && ;0 < t \le \tau \\ U(L,t) &= g_1(t) && ;0 < t \le \tau \end{aligned} \tag{28}$$

In equation (17), $U(x, 0)$ is the initial condition while $g_0(t)$ and $g_1(t)$ are boundary conditions.

The heat conduction equation (27) above is discredited by the Difference Method so that it is obtained

$$x_{k+1} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \\ u_N \end{bmatrix}_{k+1} = \begin{bmatrix} 1-2p & p & 0 & 0 & \dots & 0 \\ p & 1-2p & p & 0 & \dots & 0 \\ 0 & p & 1-2p & p & \dots & 0 \\ 0 & 0 & p & 1-2p & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1-2p \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \\ u_N \end{bmatrix}_k + \begin{bmatrix} pu_0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

with $U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}$, $u_t = u(x_t, t)$, heat at position $x = t$ at t with $t = 1,2,3,\dots,N$.

Furthermore, we get:

matrix $\mathbf{A} = \begin{bmatrix} 1-2p & p & 0 & 0 & \dots & 0 \\ p & 1-2p & p & 0 & \dots & 0 \\ 0 & p & 1-2p & p & \dots & 0 \\ 0 & 0 & p & 1-2p & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1-2p \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} pu_0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

Suppose the observation of the temperature of the wire rod is carried out in positions $N-3$ and $N-2$. This means that observations are made on u_{N-3} and u_{N-2} so that the output vector $z(t)$ can be written as

$$z_k = \mathbf{C}x_k,$$

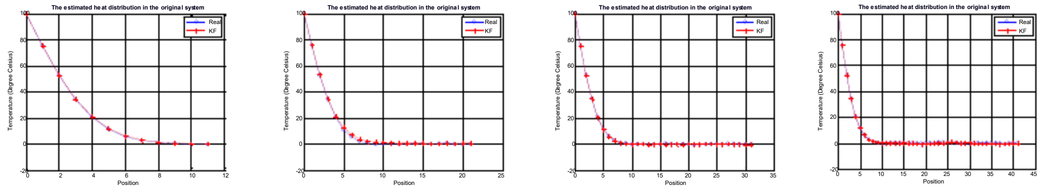
with $z(t) \in \mathbb{R}^2$ and $\mathbf{C} \in \mathbb{R}^{2\times N}$ with $\mathbf{C}(1, N-3) = 1$, $\mathbf{C}(2, N-2) = 1$, and $\mathbf{C}(i,j) = 0$ for another i,j.

By taking $\mathbf{D} = 0$, then we get the $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ system for the heat conduction problem.

3.2. Simulation Results

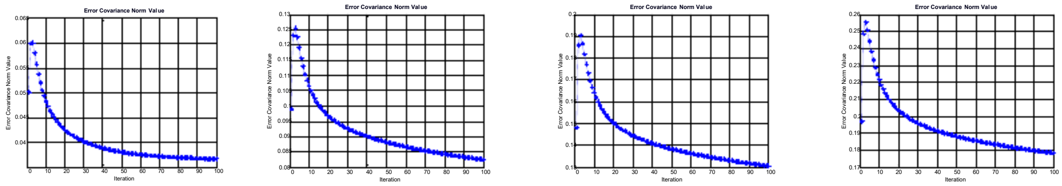
In this simulation, the length of the metal rod is 100 cm, the heat conduction coefficient is $\alpha = 0{,}05\ \frac{\text{kkal}}{\text{s.m}}{}^{\text{o}}C$, boundary conditions are $U_0 = 100$ and $U_N = 0$. The initial values of the covariance error are $P_0 = 0{,}01$, $\hat{x}_0 = 0$, $Q = 0{,}01$, and $R = 0{,}01$. The number of iterations is done as much as $T = 100$, and $N = 10$, $N = 20$, $N = 30$, and $N = 40$ are taken.

In this simulation, we compare the estimated results of the KF algorithm for the initial system and the reduced system with the LMI method.



a. $N = 10$ b. $N = 20$ c. $N = 30$ d. $N = 40$

Figure 2 Estimated heat distribution using the KF in the initial system for $N = 10$, $N = 20$, $N = 30$, and $N = 40$.



a. $N = 10$ b. $N = 20$ c. $N = 30$ d. $N = 40$

Figure 3 The error covariance norm value for the initial system for $N = 10$, $N = 20$, $N = 30$, and $N = 40$.

Model reduction using BT method is done by cutting variables in a balanced system that is difficult to control and difficult to observe, namely state variables that correspond to the singular value of small Hankel. The Hankel singular value sequence starts from large to small, namely $\sigma_1 \geq \cdots \geq \sigma_r \geq \sigma_{r+1} \geq \cdots \geq \sigma_n \geq 0$. Based on the Hankel singular value sequence, BT method can be done by cutting the Hankel singular value to the r-order where $\sigma_r \geq \sigma_{r+1}$ and applies :

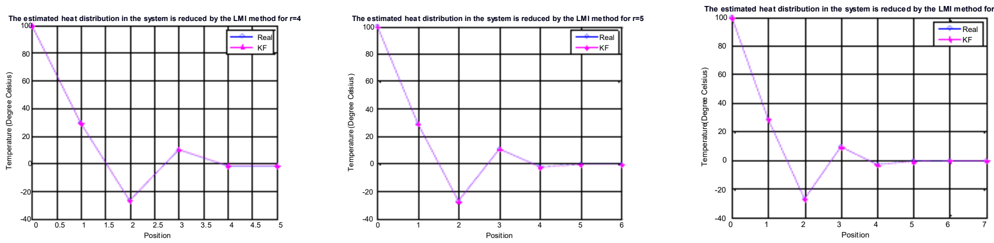
$$\|G(z) - G_r(z)\|_\infty \leq 2(\sigma_{r+1} + \cdots + \sigma_n).$$

The BT method guarantee the upper bounds of the error $\| G(z) - G_r(z)\|_\infty$ is $2(\sigma_{r+1} + \cdots + \sigma_n)$, and then using the LMI method, we can find the supremum of the error $\| G(z) - G_r(z)\|_\infty$, denoted by nonnegative scalar γ which satisfy $\sigma_{r+1} \leq \|G(z) - G_r(z)\|_\infty < \gamma \leq 2(\sigma_{r+1} + \cdots + \sigma_n)$.

Table 1 Comparison of the error $\|G(z) - G_r(z)\|_\infty$ obtained by the LMI method, BT method, and the upper bounds of the error reduction.

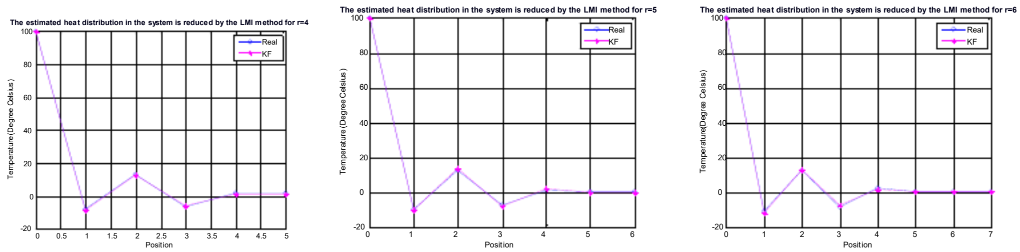
N		**Lower bounds of error reduction**	**Error reduction model with LMI method**	**Error reduction model with BT method**	**Upper bounds of error reduction**
	r	$\boldsymbol{\sigma_{r+1}}$	$\boldsymbol{\gamma}$		$\boldsymbol{2(\sigma_{r+1} + \cdots + \sigma_n)}$
10	4	0.04256400	0.045	0.076845	0.09931255
	5	0.00566750	0.007	0.010354	0.01418455
	6	0.00127960	0.0014	0.00230390	0.00284955
20	4	0.0173410	0.019	0.030942	0.042492262399053
	5	0.0032090	0.004	0.00572170	0.007810262399053
	6	0.000594560	0.0009	0.00104850	0.001392262399053
30	4	0.0144050	0.03	0.0256740	0.033815615
	5	0.00210940	0.003	0.00378990	0.005005615
	6	0.000326390	0.0004	0.00058253	0.000786815
40	4	0.0122780	0.02	0.0218070	0.028847843
	5	0.00183280	0.003	0.00329920	0.004291843
	6	0.000266450	0.0006	0.00048043	0.000626243

Table 1 shows that the error of model reduction using the LMI method is smaller than error of model reduction using the BT method and the upper bounds of the error reduction. Furthermore, we obtain that system that have been reduced by the LMI method, for $r = 4, 5, 6$ is a stable, controlled, and observed systems. We obtain the estimated heat distribution in the system is reduced by the LMI method for $r = 4, 5, 6$ as follow :



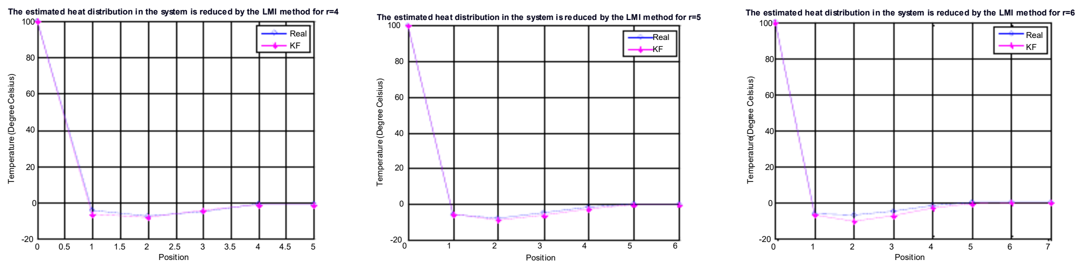
$r = 4$ $r = 5$ $r = 6$

Figure 4 The estimated heat distribution in the system is reduced by the LMI method for $N = 10$



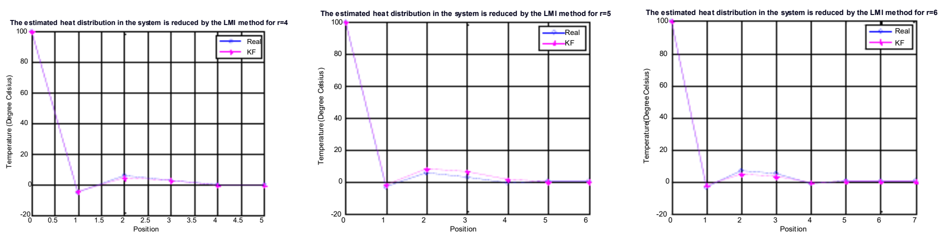
$r = 4$ $r = 5$ $r = 6$

Figure 5 The estimated heat distribution in the system is reduced by the LMI method for $N = 20$



$r = 4$ $r = 5$ $r = 6$

Figure 6 The estimated heat distribution in the system is reduced by the LMI method for $N = 30$



$r = 4$ $r = 5$ $r = 6$

Figure 7 The estimated heat distribution in the system is reduced by the LMI method for $N = 40$

Table 2 The norm covariance error value by using the Kalman filter algorithm

Initial system		**Reduction model by LMI method**	
State variable $\boldsymbol{N}$	Average of error covariance norm	State variable $\boldsymbol{r}$	Average of error covariance norm
40	0.193897759220517	4	0.555528120109958
		5	0.616010364313372
		6	0.653012441427901

From Figures 2, it can be seen that the implementation of the KF algorithm in the initial system produces a very good estimation because the plot of the estimated variable results are very close to the real state variable. This is also strengthened by the results of the plot of the error covariance which show that the results of the error covariance are convergent, shown in Figure 2. From Figures 4, 5, 6, and Figure 7, it can be seen that the implementation of the KF algorithm in the reduced system produces a very good estimation too because the plot of the estimated variable results are very close to the real state variable. From Table 2, it can be seen that the error covariance value of the estimated heat distribution in the reduced system for $N = 40$ is close to zero even though the error covariance norm value in the initial system is actually smaller than the reduced system.

4. CONCLUSION

From the results of the analysis above, we obtain that the error of model reduction using the LMI method was smaller than the error of model reduction using the BT method and the upper bounds of the error reduction. Furthermore, we obtain that system that have been reduced by the LMI method, for $r = 4, 5, 6$ is a stable, controlled, and observed systems. We also obtain that implementation of the KF algorithm in the initial system and the system that have been reduced by the LMI method are a very good estimation approaching the real state variable. This is confirmed by the norm covariance error whose value converges to near zero. It is obtained that the error covariance value of the estimated heat distribution in the reduced system for $N = 40$ is close to zero even though the error covariance norm value in the initial system is actually smaller than the reduced system.

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