ON PERIODIC MOTION OF SIMPLE PENDULUM: "AND YET, IT MOVES."

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Abstract

For solving the nonlinear differential equation of the pendulum, here we adopt a method that transforms the nonlinear differential equation into an equivalent linear one and then evaluate the period oscillation. We also apply the energy conservation principle to find the dependence of the time period on the amplitude of oscillation. Also harmonic balance method is applied to find an expression for the period of oscillation. Theoretical curves, simulation results and experimental results are given in support of the findings. Nonlinear method 1 proves that the pendulum oscillation is periodic but it has also very small amount of third harmonic. FFT analysis has been carried out of the data as obtained from the simulation results and it is found that system is almost free from harmonic distortion.

1. Historical Note:

At age twenty, Galileo Galilei (15 February 1564 - 08 January 1642) noticed around 1602 a lamp swinging back and forth while he was in a cathedral of Pisa. He used his pulse to time large and small swings. Galileo discovered something that no one else had ever realized: the period of each swing was exactly the same, i.e. "The law of Isochronisms" and physics of oscillation was born. Isochronous is a word derived from the Greek iso (equal) and chronos (time). Galileo later established that the period of a pendulum varies as the square root of its length and is independent of the material of the pendulum bob (the mass at the end). Pendulums are mentioned in both Galileo's Dialogue “Concerning the Two Chief World Systems” and his “Dialogues Concerning Two New Sciences”. In these two works, Galileo discusses some of the major points he discovered about pendulums. In 1673, Christiaan Huygens published his theory of the pendulum, where it is shown that for small oscillation the time period ‘T₀’ is given by
\[
T_0 = 2\pi \sqrt{\frac{L}{g}}
\]

\(T_0\) is known as Christiaan Huygens's law for the period

2. Introduction:

A simple pendulum consists of a particle of mass ‘m’ hanging from an unstretchable string of length L, fixed at a frictionless pivot point P. This is shown in Fig.1. When a pendulum is displaced sideways from its resting equilibrium position, it is subject to a restoring force due to gravity that will accelerate it back toward the equilibrium position. The pendulum will swing back and forth with periodic motion in a uniform constant gravitational field. Incidentally, the word 'pendulum' is from the New Latin, ‘pendulus’, meaning ‘hanging’. Mathematicians would describe the pendulum as a system that only exhibits one degree of freedom.

Pendulums are used to regulate pendulum clocks, and are used in scientific instruments such as accelerometers and seismometers. Historically they were used as gravimeters to measure the acceleration of gravity in geophysical surveys, and even as a standard of length. Another interesting application is called the Foucault pendulum. This pendulum will demonstrate the Earth’s rotation.

When one studies the motion of a simple pendulum we come across two terms like “Isochronous” and “Simple Harmonic Motion”[1]. The meaning these two are interlinked. When a particle is displaced from its equilibrium position, like that shown in Fig.1, it is subjected to restoring force which is always directed towards the equilibrium position and as a result the particle executes back-and-forth motion. When the restoring force is proportional to the displacement and acts along the line of displacement we call the motion executed under this condition as simple harmonic motion. If now refer to the pendulum motion, we find that any instant the displacement is \(x = L \sin \phi\) and the restoring force is \(F = mg \sin \phi\). Therefore,

\[
\frac{F}{x} = \frac{mg}{L},
\]

which is constant. But, the direction of the restoring force is not in line with the displacement and therefore the motion of the pendulum is not a simple harmonic one.

Referring to Fig.1 it is easy to show that the motion of the simple pendulum is governed by the following Newton’s law of motion

\[
\text{mass} \times \text{acceleration} = \text{restoring force}
\]

\[
mL \frac{d^2 \phi}{dt^2} = -mg \sin(\phi)
\]
This is the Pendulum Equation, which is a nonlinear differential equation with a periodic nonlinearity, namely, $\sin(\phi)$, which, however, can be equivalently represented by a cubic type nonlinearity operating within an angle of swing $\pm \pi / 2$.

$$\sin(\phi) = 0.990\phi - 0.140\phi^3 \quad -2 \leq \phi \geq +2$$  \hspace{1cm} (2)

Referring to the plot as shown in Fig.2, the approximation the periodic nonlinearity by non-periodic one is very close for the range in question. Once this is done the pendulum equation transforms into a Duffing’s equation

$$\frac{d^2\phi}{dt^2} + \alpha \phi - \gamma \phi^3 = 0.$$  \hspace{1cm} (3)

The nonlinear differential equation for the simple pendulum can be exactly solved and the period and periodic solution expressions involve the complete elliptic integral of the first kind and the Jacobi elliptic functions, respectively [2-4]. Due to this, several approximation schemes have been developed to investigate the situation for large amplitude oscillations of a simple pendulum [2].

**Numerical Experiment:**

In order to justify the validity of equivalence of the two equations (1) and (3) numerical simulations were carried out using MATLAB and MATCAD. The results when the pendulum executes oscillations with large amplitude of $\pi / 3$ radians for both the representations of the pendulum equation clearly shows identical behaviour with a free running oscillation frequency of 0.94 Hz, where as for very small angle oscillation the frequency is 1.0 Hz. These are illustrated in Figure 3 and Figure 4. It is also observed from the numerical experiment that the oscillation is almost sinusoidal/cosinusoidal.

Possibly Galileo did not consider the dependence of the period of oscillation on the maximum angle of displacement “$\phi_0$”. However, later on the researchers found that the time period $T$ is given as

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi / 2} \frac{d\phi}{\sqrt{1-k^2 \sin^2 \phi}}; \text{where } k = \sin(\phi_0 / 2).$$  \hspace{1cm} (5)

The integral is known as the elliptical integral of first kind which is tabulated in the tables of integral. The period is also written in the series form:[3,4]

$$T = 2\pi \sqrt{\frac{L}{g}} \left(1 + \frac{\phi_0^2}{16} + \frac{11}{3072} \phi_0^4 + \frac{173}{737280} \phi_0^6 + \frac{22931}{132105760} \phi_0^8 + \cdots \right).$$  \hspace{1cm} (6)

The relations as given in (5) and (6) are neither simple nor closed form expressions.
In contradiction to the earlier articles written on the subject of simple pendulum our purpose is two-fold:

1. To consider the differential equation of the motion of the pendulum considering very small angle of swing when we replace sine of an angle by the angle itself, i.e., sin $\theta \equiv \theta$. Here we solve for the period of oscillations through the following methods:
   a. Conventional Method
   b. Conservation of Energy Principle
   c. Geometrical Method
   d. Algebraic Method

2. We consider the nonlinear differential equation of the pendulum. Here we adopt a method that transforms the nonlinear differential equation into an equivalent linear one and then evaluate the period oscillation. We also apply the energy conservation principle to find the dependence of the time period on the amplitude of oscillation. Also harmonic balance method is applied to find an expression for the period of oscillation. Theoretical curves, simulation results and experimental results are given in support of the findings.

3.1 Linear Method

3.1.1 Conventional Method:

We begin with the linear pendulum equation, i.e. when we replace sin $\phi$ by its angle ($\phi$) only, '$\phi$' is usually assumed to be less than four degree ($\frac{\pi}{3}$ radians). Thus the pendulum equation (1) approximates to

$$\frac{d^2 \phi}{dt^2} + \omega_0^2 \phi = 0,$$  \hspace{1cm} (6a)

with the initial condition,

$$t = 0, \phi = \phi_0 \text{ and } \frac{d\phi}{dt} = 0.$$  \hspace{1cm} (7)

Multiply both sides by $2 \frac{d\phi}{dt}$ and integrate

$$\left( \frac{d\phi}{dt} \right)^2 = \omega_0^2 \left( \phi_0^2 - \phi^2 \right)$$

Applying the initial conditions and put $\phi = \phi_0 \cos \omega_0 t$ and integrate to find

$$\phi = \phi_0 \cos \omega_0 t$$

Thus the period of oscillation is

$$T = \frac{2\pi}{\omega_0} = 2\pi \frac{L}{g}.$$  \hspace{1cm} (8)
3.1.2 Energy Method:

When the pendulum is oscillating, the energy is partly kinetic and partly potential. The kinetic energies (KE) is stored in the mass by virtue of its velocity where as potential energy is stored by virtue of its position, i.e. work done against the gravity. The total energy is constant. So we can write

\[
[KE + PE]_{\text{Position}_1} = [KE + PE]_{\text{Position}_2}. \tag{9}
\]

Let the position 1 denotes the point of the static equilibrium, whereas the position2 denotes the point of maximum deflection \( \phi = \phi_0 \). Thus

\[
[KE]_1 = [PE]_2.
\]

Because at the position 1, PE=0 and at the position 2, KE=0

Let as assume, \( \phi = \phi_0 \cos \omega t \)

\[
KE = \frac{1}{2} m \left[ L \frac{d\phi}{dt} \right]^2 \tag{10}
\]

\[
PE = mg(L - L \cos \phi) \tag{11}
\]

Thus

\[
\frac{1}{2} \left( L \phi_0 \omega \right)^2 = gL \sin^2 \left( \frac{\phi_0}{2} \right)
\]

\[
\omega = \sqrt{\frac{g}{L}} \sin \left( \frac{\phi_0}{2} \right) \tag{12}
\]

when \( \phi_0 \) is small, \( \omega = \sqrt{\frac{g}{L}} = \omega_0 \). \tag{13}

3.1.3 Geometrical Method:

Referring to (2), (10) and (11), it is seen that

\[
\frac{1}{2} m \left( L \frac{d\phi}{dt} \right)^2 + mgL(1 - \cos \phi) = \text{Const.}
\]

\[
\left( \frac{d\phi}{dt} \right)^2 + 4 \frac{g}{L} \sin^2 \left( \frac{\phi}{2} \right) = \text{Const.} \tag{14}
\]

When \( \phi \) is small, \( \sin \frac{\phi}{2} \leq \frac{\phi}{2} \)

\[
\left( \frac{d\phi}{dt} \right)^2 + \frac{g}{L} \phi^2 = \text{Const.}
\]

Or \( \left( \frac{\omega}{\omega_0} \right)^2 + \phi^2 = \text{const.} \)

That is \( \left( \frac{\dot{\phi}}{\omega_0} \right)^2 + \phi^2 = R^2 \). \tag{15}

Using the initial condition, namely, at \( t = 0, \phi = \phi_0, \dot{\phi} = 0 \) we find

\[
R^2 = \phi_0^2 \tag{16}
\]
Equation (15) is an equation of circle in the $\phi/\omega_0$ and $\phi$ plane, as depicted in the figure 5. Triangles NOP and NAB are similar with $\angle NAB = \angle NOP$, $\angle ANB = \angle ONP$. Thus

$\frac{NB}{NA} = \frac{NP}{ON}$  \hspace{1cm} (17)

Now as the point N rotates clockwise on the circular path with an angular velocity $\omega$. So the point P executes a simple harmonic motion. The tangential velocity, $v_A = NA$, is obtained from (17) as

$\frac{\dot{\phi}}{V_A} = \frac{\dot{\phi} / \omega_0}{R}$ \hspace{1cm} i.e. $V_A = R\omega_0$  \hspace{1cm} (18)

Again, since the point N is moving on the circular path with an angular velocity $\omega$, then

$V_A = R\omega$  \hspace{1cm} (19)

Therefore, $\omega = \omega_0$  \hspace{1cm} (20)

Since, by the initial condition, at time, $t = 0$, $\phi = \phi_0$, $ON = \phi_0$ The angle is obtained as $\psi = \omega_0t$

Therefore,

$\phi(t) = OP = R\cos \omega_0t = \phi_0 \cos \omega_0t$  \hspace{1cm} (21)

3.1.4 Algebraic Method:

Refer to equation (2) and putting $x = \frac{d\phi}{dt} - p\phi$ we can write

$\frac{d\phi}{dt} + px + \phi\left(p^2 + \omega_0^2\right) = 0$  \hspace{1cm} (22)

Since, $p$ is an arbitrary constant; it can be real or imaginary. Thus we can write

$p^2 + \omega_0^2 = 0$ \hspace{1cm} (23)

i.e. $p = \pm j\omega_0$

and $\frac{dx}{dt} + px = 0$  \hspace{1cm} (24)

Once $x$ is known then $\phi$ can be evaluate from (22). Now we solve (24) algebraically. But we need to know the initial value of $x$. we have assumed that at time $t = 0$, $\phi = \dot{\phi}$, $\dot{\phi} = 0$

Therefore, $x(0) = -p\phi_0$  \hspace{1cm} (25)

Equation (26) can be written as

$\frac{x(t + \Delta t) - x(t)}{\Delta t} = -px(t)$

$x(t + \Delta t) - x(t) = -p\Delta tx(t)$  \hspace{1cm} (26)

i.e. $x(\Delta t) = -p\Delta t.x(0) + x(0) = -(1 - p\Delta t).p\phi_0$

$x(2\Delta t) = x(\Delta t)[1 - p.\Delta t]\]

$= -(1 - p\Delta t)^2 p\phi_0$

$x(n\Delta t) = -(1 - p\Delta t)^n p\phi_0$  \hspace{1cm} (27)

Put $p\Delta t = z$, then as $\Delta t \to 0$, $z \to 0$, $[n\Delta t]_{\Delta t \to 0} \to t$

Therefore,
\[
\frac{x(t)}{\phi_0} = (1-z)^n = (1-z)^{\frac{m}{z}} = \left[(1-z)^{\frac{1}{z}}\right]^m
\]
\[
= \left[\frac{1}{(1+z)^{\frac{1}{z}}}\right]^{pn\Delta t}
\]
(28)

As \( \Delta t \to 0 \), and \( n \to \infty \), \( z \to 0 \)
\[
\frac{x(t)}{-p\phi_0} = \left[\frac{1}{e}\right]^{pt} = e^{-pt}
\]
(29)

Use is made of Starling approximation.
\[
\lim_{z \to 0} \left[\frac{1}{(1+z)^{\frac{1}{z}}}\right] = e \quad \text{or,} \quad x(t) = -p\phi_0 e^{-pt}
\]
(30)

Therefore, from (22) and (29)
\[
\frac{d\phi}{dt} - p\phi = -\phi_0 e^{-pt}
\]
(31)

Put \( \psi = \phi e^{-pt} \) in (33) and write as
\[
\frac{d\psi}{dt} = -p\phi_0 e^{-2pt}
\]
(32)

It is easily seen that \( \psi(0) = \phi_0 \). Thus we express (32) as
\[
\psi(t+\Delta t) = \psi(t) - p\phi_0 e^{-2pt}
\]

Preceding exactly in a similar way it is not difficult to show that
\[
\psi(n\Delta t) = \phi_0 - \phi_0 p\Delta t \left(1 + e^{-2p\Delta t} + e^{-4p\Delta t} + e^{-6p\Delta t} + \cdots + e^{-2(n-1)p\Delta t}\right)
\]
(33)

Now as \( n \to \infty \), and \( \Delta t \to 0 \); then \( n\Delta t = t \)
Therefore, equation (33) turns out
\[
\phi = \phi_0 \left(e^{pt} - e^{-pt}\right)
\]
(34)

Which can be solved easily and the solution given by
\[
\phi(t) = \phi_0 \cos \omega_{\phi} t
\]
(34a)

### 3.2.1 Non linear Method 1

Refer to equation (1) and substitute
\[
y^2 = 2\int_0^\phi \sin \phi \, d\phi
\]
(35)
i.e.
\[
y = \sin \left(\frac{\phi}{2}\right)
\]
(36)
And \[ \frac{dy}{d\phi} = \sin \phi \quad (37) \]

It is interesting to see that \( y \equiv \frac{\pi}{2} \) for \( 0 \leq \frac{\pi}{2} \leq \frac{\pi}{2} \) with an error 4.7\%. That is \( \phi \) can be taken to \( \frac{\pi}{2} \) for linear analysis through this conversion.

Referring to (1) and replace \( t \) to \( \tau \) and \( y \) to \( \phi \) simultaneously one can show

\[ \frac{d^2 \phi}{dt^2} = \frac{d^2 y}{d\tau^2} \left[ \frac{d\phi}{dy} \cdot \frac{d\tau}{dt} \right] \frac{d\tau}{dt} \quad (38) \]

Again assume

\[ \frac{dy}{d\phi} = \frac{d\tau}{dt} \quad (39) \]

That is \( \frac{d\tau}{dt} = \sin \phi \quad (40) \)

Using (1), (37), (38) and (40)

One finds that

\[ \frac{d^2 y}{d\tau^2} + \alpha_0^2 y = 0 \quad (41) \]

It is seen that when \( t = 0, \tau \) is also zero. Thus

\[ y(\tau = 0) = y_0 = 2 \sin \frac{\pi}{2} \quad \text{(cf.36)} \quad (42) \]

Thus \( y(\tau) = 2 \sin \left( \frac{\pi}{2} \right) \cos (\alpha_0 \tau) \quad (43) \)

From (40) and (43)

\[ \frac{d\tau}{dt} = \frac{\sin(y_0 \cos \alpha_0 \tau)}{y_0 \cos \alpha_0 \tau} \]

\[ = \frac{2J_1(y_0)}{y_0} - \frac{2J_3(y_0)}{y_0} \frac{\cos(3\alpha_0 \tau)}{\cos \alpha_0 \tau} \]

\[ = \frac{2J_1(y_0)}{y_0} - \frac{2J_3(y_0)}{y_0} \left[ 4 \cos^2(\alpha_0 \tau) - 3 \right] \]

\[ = \frac{2J_1(y_0)}{y_0} + \frac{2J_3(y_0)}{y_0} - \frac{4J_3(y_0)}{y_0} \cos(2\alpha_0 \tau) \quad (44) \]

Note that the maximum value of \( y \) is

\[ \left[ y_0 \right]_{\text{max}} = 2 \sin \left[ \frac{\pi}{2} \right] = \sqrt{2} \quad (45) \]

It is seen that

\[ J_1(\sqrt{2}) = 0.504, \quad J_3(\sqrt{2}) = 0.052, \quad J_3(2) = 1.355 \times 10^{-3} \]

Therefore

\[ \frac{d\tau}{dt} \equiv \frac{2}{y_0} \left[ J_1(y_0) + J_3(y_0) \right] - \frac{4J_3(y_0)}{y_0} \cos 2\alpha_0 \tau \quad (46) \]

For convince we put

\[ a = \frac{2}{y_0} \left[ J_1(y_0) + J_3(y_0) \right] \quad \text{and} \quad b = \frac{4J_3(y_0)}{y_0} \]

\[ \left[ \frac{a}{b} \right]_{\text{max}}(y_0) = \frac{J_1(\sqrt{2}) + J_3(\sqrt{2})}{2J_3(\sqrt{2})} = 205 \quad (47) \]
Putting \( \frac{a}{b} = x \), one finds from (46)

\[
\frac{b \, dt}{d\tau} = \frac{1}{x - \cos 2\omega_0 \tau} = 1 + 2 \sum r^n \cos(2n\omega_0 \tau) \sqrt{x^2 - 1}
\]

(48)

Where \( r = x - \sqrt{x^2 - 1} \)

(49)

from (48) using method of sinesure approximation one finds

\[
\omega_0 \tau \approx \omega_0 \frac{b \sqrt{x^2 - 1}}{y_0} t - \frac{r}{y_0} \sin \left( \frac{2b}{y_0} \sqrt{x^2 - 1} \omega_0 t \right)
\]

(50)

That is

\( \omega = \text{radian frequency of oscillator} \)

\[
\omega = \frac{\omega_0 b \sqrt{x^2 - 1}}{y_0}
\]

\( \approx 2\omega_0 \left[ J_1(y_0) + J_3(y_0) \right] \)

(51)

That is

\[
\frac{T}{T_0} = \frac{y_0}{2 \left[ J_1(y_0) + J_3(y_0) \right]}
\]

(52)

Where, \( y_0 = 2 \sin \left( \frac{\phi_0}{2} \right) \)

Equation (50) is rewritten as

\[
\omega_0 \tau = \omega t - m \sin 2\omega t
\]

(53)

Where \( m = \frac{r}{y_0} \)

Thus \( \cos(\omega_0 \tau) = \cos[\omega t - m \sin(2\omega t)] \)

(54)

Since ‘m’ is very small,

\[
\cos \omega_0 \tau \approx \cos \omega t + \sin \omega t \cdot m \sin 2\omega t\]

\[
= \cos \omega t + \frac{m}{2} \left[ \cos 3\omega t - \cos \omega t \right]
\]

\[\approx \cos \omega t + \frac{m}{2} \cos 3\omega t \]

\[y = 2 \sin \left( \frac{\phi}{2} \right)\]

Thus \( \phi \approx y = 2 \sin \left( \frac{\phi_0}{2} \right) \left[ \cos \omega t + \frac{m}{2} \cos 3\omega t \right] \)

(55)

Thus it is seen that the pendulum oscillation is periodic but it has also very small amount of third harmonic.

### 3.2.2 Energy Balance Principle for Large Amplitude

Let us assume that

\[
\phi = y_0 \cos \omega t
\]

(56)
Using (56), (10) and (11) one finds

\[ \frac{1}{2} \left( L y_0 \omega \right)^2 = 2 g L \sin^2 \left( \frac{\phi_0}{2} \right) \]

\[ \omega^2 = \frac{4 \omega_0^2 \sin^2 \left( \frac{\phi_0}{2} \right)}{y_0^2} \]

\[ \omega = 2 \omega_0 \frac{\sin \left( \frac{\phi_0}{2} \right)}{y_0} \rightarrow \omega_0 \frac{\sin \left( \frac{\phi_0}{2} \right)}{\left( \frac{\phi_0}{2} \right)} \]

\[ \frac{T}{T_0} = \frac{\phi_0}{2 \sin \left( \frac{\phi_0}{2} \right)} \]

(57)

The departure from the assumption of Isochronisms is shown in the Figure 6.

### 3.2.3 Harmonic Balance Method:

We have already seen that the oscillation of the pendulum is nearly sinusoidal only a very small amount of third harmonic distortion is present which for practical purpose may considered negligible. That is, we assume that oscillation of the pendulum can be represented as

\[ \phi = \phi_0 \cos (\omega \tau) \]

Therefore,

\[ \sin \phi = 2 J_1 (\phi_0) \cos (\omega \tau) - 2 J_3 (\phi_0) \cos (3\omega \tau) \]

Again: \( \frac{d^2 \phi}{dt^2} = -\phi_0 \omega^2 \cos (\omega \tau) \)

(58)

Using the pendulum equation and (58) it can be easily shown by equating the coefficient of the fundamental terms

\[ \omega = \omega_0 \sqrt{\frac{J_1 (\phi_0)}{\phi_0 / 2}} \]

(59)

\[ \frac{T}{T_0} = \sqrt{\frac{\phi_0 / 2}{J_1 (\phi_0)}} \]

(60)

The departure from the assumption of Isochronisms is shown in the Figure 6.

**Conclusion:**

Figure 6 shows the departure from the theory of isochronisms even when the maximum angle swing is 4° (i.e. 0.07 radians). It demonstrates that a simple pendulum never executes simple harmonic motion. From nonlinear method 1 it is seen that the pendulum oscillation is periodic but it has also very small amount of third harmonic. FFT analysis has been carried out of the data as obtained from the simulation results and it is found that system is almost free from harmonic distortion. The result is shown in the Figure 4. From the pendulum motion, we find that
any instant the direction of the restoring force is not in line with the displacement (Figure 1) and therefore the motion of pendulum is not a simple harmonic one.

Acknowledgement

Authors are thankful to the management of Sir J.C. Bose School of Engineering for carrying out the work at Sir J.C Bose Creativity Centre of Supreme Knowledge Foundation Group of Institutions, Mankundu, Hooghly.

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