BUILDING A MODEL FOR EXPECTED COST FUNCTION TO OBTAIN DOUBLE BAYESIAN SAMPLING INSPECTION

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ABSTRACT

This paper deals with building a model for total expected quality control cost for double Bayesian sampling inspection plan, which consist of drawing a sample \((n_1)\) from a lot \((N)\), then checking the number of defective \((x_1)\) in \((n_1)\), if \((x_1 \leq a_1)\), were \((a_1)\) is acceptance number, then accept and stop, if \((x_1 \geq r_1)\), were \((r_1)\) is rejection number, then take a decision to reject and stop sampling, if \((a_1 < x_1 < r_1)\), draw a second sample \((n_2)\) and check the number of defective \((x_2)\) in \((n_2)\), if \((x_1 + x_2 \leq a_2)\), accept and stop, if \((x_1 + x_2 \geq r_2)\) reject and stop. The model derived assuming that the sampling process is Binomial with \((n,p)\) and \((p)\) is random variable, varied from lot to lot and have prior distribution \(f(p)\), here it is found Beta prior with parameters \((S,T)\). The model is applied on the set of real data obtained from some industrial companies in Iraq, were the percentage of defective of \((140)\) lots are studied and tested, \(f(p)\) is found Beta prior \((S,T)\). These parameters are estimated by moment estimator. After checking the distribution \(f(p)\) and all the parameters of quality control model, we work first on obtaining single Bayesian sampling inspection plan \((n_1, c_1)\) from minimizing the standard regret function \((R_1)\), then we find the parameters of double Bayesian sampling inspection plan from minimizing \((R_2)\), for various values of process average of quality \((\overline{p})\), with different lot size \(N\).

Keywords: Bayesian sampling Inspection, prior distribution, average of quality, standard regret function for single sampling plan. standard regret function for double Bayesian sampling plan.
1- AIM OF RESEARCH

The aim of this research is to build a model for total cost of double Bayesian sampling plans, then determine the parameters of this double Bayesian sampling plans \((n_1, a_1, r_1, n_2, a_2, r_2)\) from minimizing the total expected cost function.

2- NOTATIONS

Here we introduce all necessary notations:

- \(n_1\) : First sample size drawn from lot N.
- \(x_1\) : Number of defective units in sample \((n_1)\).
- \(a_1\) : Number of accepted defective units in \((n_1)\).
- \(r_1\) : Number of rejected units in \((n_1)\).
- \(n_2\) : Size of second sample taken from \((N - n_1)\).
- \(x_2\) : Number of defective units in \((n_2)\).
- \(X\) : Number of defective units in lot \((N)\).
- \(a_2\) : Number of accepted defectives in \((n_2)\).
- \(r_2\) : Number of rejected units second sample \((n_2)\).
- \(P\) : Percentage of defectives in production.
- \(p\) : percentage of defectives in sample.
- \(P_a^{(1)}(p)\) : Probability of accepting the product with quality \((p)\) according to the decision from first sample \((n_1)\).
- \(P_r^{(1)}(p)\) : Probability of rejecting the product with quality \((p)\) due to the decision of rejecting from first sample \((n_1)\).
- \(P_a^{(2)}(p)\) : Probability of accepting the product with quality \((p)\) according to the decision from second sample \((n_2)\).
- \(P_r^{(2)}(p)\) : Probability of rejecting the product with quality \((p)\) due to the decision of rejecting from first sample \((n_2)\).
- \(P_s^{(2)}(p)\) : Probability of continue sampling after the first sample is chosen and the value of \((a_1 < x_1 < r_1)\), so we cannot take a decision to accept or reject according to first sample.
- \(kS(p)\) : Average inspection cost from first sample, where;
  \(E(n_1S_1 + x_1S_2) = n_1(S_1 + S_2P) = n_1kS(p)\)
- \(S_1\) : Cost of inspecting item from first sample.
- \(S_2\) : Cost of repairing or replacement of item in sample \((n_1)\).
- \(R_1\) : Cost of sampling and testing unit in rejected quantity \((N - n)\).
- \(R_2\) : Cost of repairing or replacement unit in rejected quantity \((N - n)\).
- \(A_1\) : Cost of accepting good units which not represents penalty cost.
- \(A_2\) : Cost of accepting defective units.

3. INTRODUCTION

In rectifying inspection plans, each item produced by a process is inspected according to some plan and then according to the number of defectives units in the sample, the lot is either accepted or the lot is rejected. Gunether [1977] explained how to obtain the parameters \((n, c)\) for single sampling plan for ordinary sampling and for Bayesian sampling, using the model of expected total quality control cost depends on cost of inspection and sampling and
cost due to wrong decision, which is accepting bad lot and rejecting good one. Another researchers gives model for obtaining parameters of single sampling inspection plan \((n, c)\) under truncated time to failure, see Epstein, B. [1962]. Goode and Kao [1960]. Gupta, S. S. [1962] who designed sampling inspection plans for normal and lognormal distribution according to pre – determined level of percentage of defectives. Also Gupta and Kunda [1999] discuss the properties of generalized exponential distributions and how to use it is constructing Bayesian sampling. Baklizi, A. [2003] introduced how to obtain acceptance sampling plans based on truncated life tests in the Pareto distribution of second kind. The research in this subject were developed when Balakrishnan, N. [2007] designed a group of sampling plans based on truncated time to failure, see Epstein, B. [1962]. Goode and Kao [1960]. Gupta, S. S. [1999] discuss the properties of generalized exponential distributions and how to use it is constructing Bayesian sampling. Baklizi, A. [2003] introduced how to obtain acceptance sampling plans based on truncated life tests in the Pareto distribution of second kind. The parameters of single sampling inspection plans \((n, c)\) are obtained from minimizing total expected quality control cost.

Now, in this paper we introduce (a mixed – Binomial) quality control expected cost function, to obtain the parameters of double Bayesian sampling inspection plans \((n_2, c_2)\) for continuous prior distribution. The parameters of single sampling attribute plans for generalized Birnbaum – Saundres distribution. In [2012], Dhwyia S. Hassun, et, built a model for total expected quality control cost, and compared it with the model of Schmidt – Taylor [1973]. Also Hald, A. [1968] gives a model for Bayesian single sampling inspection sampling for generalized Birnbaum – Saundres distribution. In [2012], Dhwyia S. Hassun, et, built a model for total expected quality control cost, and compared it with the model of Schmidt – Taylor [1973]. Also Hald, A. [1968] gives a model for Bayesian single sampling inspection sampling for generalized Birnbaum – Saundres distribution.

Building Model of Total Expected Cost Function

The model comprises:

\[ a- ks(p) : \text{Average inspection cost from first sample;} \]

\[ ks(p) = E(n_1S_1 + x_1S_2) \]

\[ = n_1(S_1 + S_2p) \]

\[ = n_1ks(p) \] \hspace{1cm} \text{ .......................................................... (1)}

\[ ks[p(x_1)] : \text{Average cost of inspecting units of second sample, under sampling distribution} \]

\[ ks[p(x_1)] = E(n_2S_1 + x_2S_2|x_1) \]

\[ = n_2(S_1 + S_2p(x_1)) \]

\[ = n_2ks[p(x_1)] \] \hspace{1cm} \text{ .......................................................... (2)}

And;

\[ p(x_1) = \sum_{xy} pr(x_1, x_2 + x_3 = y) \]

\[ = g_{n_1}(x_1) \] \hspace{1cm} \text{ .......................................................... (3)}

or;

\[ p(x_1) = \sum_{xy_2} g_{n_1, n_2}(x_1, x_2) \]

\[ = g_{n_1}(x_1) \] \hspace{1cm} \text{ .......................................................... (4)}

If we assume that \([x_3 = (x – x_1 – x_2)]\), then the joint probability distribution of \((x_1, x_2, x_3)\) is;

\[ p(x_1, x_2, x_3) = p(x_1, x_2|x_3)p(x_3) \] \hspace{1cm} \text{ .......................................................... (5)}

\[ p(x_1, x_2) = \sum_{x_3} p(x_1, x_2, x_3) \] \hspace{1cm} \text{ .......................................................... (6)}

And;

\[ p(x_2|x_1) = g_{n_2}(x_2|x_1) \] \hspace{1cm} \text{ .......................................................... (7)}

The average inspection cost for second sample is equal to:
\[ E(n_2s_1 + x_2s_2|x_1) \]
\[ = n_2(s_1 + s_2p(x_1)) \]
\[ = n_2ks[p(x_1)] \]  
\[ \text{............... (8)} \]

In case of accepting produced lot from first sample, the average cost is;
\[ E\{(N - n_1)A_1 + (X - x_1)A_2|x_1\} \]
\[ = (N - n_1)A_1 + E(x_2|x_1)A_2 \]
\[ = (N - n_1)[A_1 + A_2p(x_1)] \]
\[ = (N - n_1)ka[p(x_1)] \]

The average of acceptance cost for second sample is;
\[ E\{(N - n_1 - n_2)A_1 + (X - x_1 - x_2)A_2|x_1, x_2\} \]
\[ = (N - n_1 - n_2)ka[p(x_1, x_2)] \]  
\[ \text{............... (9)} \]

After explaining all notations, we can show that the lot size \( N \), in case of double sampling can divide into;
\[ N = n_1 + (N - n_1)\{p_a^{(1)} + p_r^{(1)}\} + n_2p_s^{(1)} + (N - n_1 - n_2)\{p_a^{(2)} + p_r^{(2)}\} \]  
\[ \text{............... (10)} \]

Assume \((km)\), is the smallest average cost, happened according to results of decisions of accepting and rejecting, where acceptance happened at \((p \leq pr)\), (i.e \(pr\), break even quality level) and rejection when \((p > pr)\), \((p)\) is random variable have \([f(p)]\) then \((km)\) is equal to;
\[ km = \int_0^{pr} (A_1 + A_2p)f(p)dp + \int_{pr}^1 (R_1 + R_2p)f(p)dp \]  
\[ \text{............... (11)} \]

Now the construction of total expected quality control cost \([k_2(p)]\) is given by;
\[ k_2 = \int k_2(p)f(p)dp \]
\[ = kns + (N - n_1) \left\{ \sum_{x_1=0}^{a_1} g(x_1)ka[p(x_1)] \right\} + \sum_{x_1=r_1}^{n_1} g(x_1)kr[p(x_1)] \]
\[ + \sum_{x_1=a_1+1}^{a_2} g(x_1)[n_2ks[p(x_1)]] \]
\[ + (N - n_1 - n_2) \left\{ \sum_{x_2=0}^{x_1} g(x_2|x_1)[ka[p(x_1, x_2)]] \right\} \]
\[ + \sum_{x_2=a_2-x_1+1}^{a_2} g(x_2)[n_2rs[p_r^{(2)}]p_r^{(2)}(x_1)f(p)kr[p(x_1, x_2)]] \]  
\[ \text{............... (12)} \]

Multiplying equation (10) by \((km)\) where \([Nkm = km]\) and apply \([k_2 - km]A_2 - R_2\), we have obtained the equation of standard cost function \((R_2)\).
\[ R_2 = n_1ds + (N - n_1) \left\{ \int_0^{pr} (pr - p)p_r^{(1)}(p)f(p)dp \right\} \]
\[ + \left\{ \int_0^{1} (p - pr)p_a^{(1)}(p)f(p)dp \right\} \]
\[ + n_2 \left\{ \int_0^{pr} (pr - p)p_s^{(1)}(p)f(p)dp + \int_0^{1} \delta(p)p_s^{(1)}(p)f(p)dp \right\} \]
\[ + (N - n_1 - n_2) \left\{ \int_0^{pr} (pr - p)p_r^{(2)}(p)f(p)dp + \int_0^{1} (p - pr)p_a^{(2)}(p)f(p)dp \right\} \]
\[ \text{............... (13)} \]
Therefore the equation of standard expected cost (Regret function) which can be simplified to:

\[ R \text{ is break even quality level; } \]

\[ pr = \frac{A_2 - R_2}{A_1 - R_1} \]

Equation (14), now transformed to standard cost function in terms of conditional distribution of percentage of defectives, which is 

\[ f(p|a_1 < x_1 < r_1) \]

especially these parts obtained from second sampling, where:

\[ f(p|a_1 < x_1 < r_1) = \frac{pr(a_1 < x_1 < r_1)f(p)}{\sum_{n=1}^{r_1} b(w(x_1, n_1))} = \frac{\{p^{(2)}(p)+p^{(2)}(p)\}f(p)}{\int_{0}^{1} \{p^{(2)}(p)+p^{(2)}(p)\}f(p)dp} \]

\[ \text{………... (15)} \]

This because when \( a_1 < x_1 < r_1 \) a second sample \( n_2 \) is drawn, and according to this sample either we accept or reject.

The final formula for standard regret function \( R_2 \) used to obtain the parameters of second sampling plan is:

\[ R_2^* = R_1(a_1, n_1, N) - (N - n_1) \sum_{x_1=a_1+1}^{r_1-1} g(x_1)dr(x_1) \]

\[ + \sum_{x_1=a_1+1}^{r_1-1} g(x_1)R_1(a_2 - x_1, n_2, N - n_1|x_1) \]  

\[ \text{………………………………. (16)} \]

Where:

\[ dr(x_1) = \int_{0}^{pr} (pr - p)f(p|x_1)dp \]

\[ R_1^*(a_1, n_1, N) = \frac{k(N,n,c) - Nkm}{ks - km} \]

\[ \text{………………………………. (17)} \]

Where:

\[ K(N, n, c) = nks + \left[ \left( A_2 - R_2 \right) \frac{S}{S + T} F_1(c, S + 1, p) \right] + \left[ (A_1 - R_1)F(c, S, p) \right] + \left[ (R_1 + R_2) \frac{S}{S + T} \right] \]

\[ ks = S_1 + S_2 \tilde{p} \]

\[ k(N, n, c) = nks + (N - n)\left[ (A_2 - R_2)\tilde{p}F_1(c, S + 1, p) \right] + \left[ (A_1 - R_1)F(c, S, p) \right] + \left[ (R_1 + R_2) \frac{S}{S + T} \right] \]

Which can be simplified to:

\[ k(N, n, c) = n(ks - km) + (N - n)\left[ (A_2 - R_2)\tilde{p}F_1(c, S + 1, p) \right] + \left[ (R_1 + R_2) \frac{S}{S + T} \right] \]

\[ + \frac{Nkm}{Nkm} \]

\[ \text{Therefore the equation of standard expected cost (Regret function } R_1 \text{) is; } \]

\[ R_1^*(N, n_1, c) = \frac{k(N,n,c) - Nkm}{ks - km} \]

Which is the first part of equation \( R_2^* \) defined in equation (16), so first of all and after determining the parameters of single Bayesian sampling plan [given in table (1)] we work on minimizing (16) to find the parameters of double Bayesian sampling inspection plan given in table (4).

Equation (16) shows that the average of standard cost function for double sampling plan, is represented by standard cost function for single sampling plan and also standard conditional cost function weighted by the probability distribution of number of defective units \( g(x_1) \), from first sample, it is obtained from;

\[ p(x_1) = g_{n_1}(x_1) = \sum_{x_2} g_{n_1, n_2}(x_1, x_2) \]
To satisfy the aim of research which is to find the parameters \((a_1, r_1, a_2)\) from minimizing \((R_2)\), we apply difference equation since the distribution of number of defectives in production is of discrete type, and also we consider:

\[
\Delta R_2 = \frac{\Delta k_2}{A_2 - R_2}
\]

The differences equations according to \((r_1, a_1, a_2)\), required to rewrite \((k_2)\) as in equation (19) follows;

Since;

\[
k_2 = k_1(a_1, n_1, N)(A_2 R_2) \sum_{x_1=a_1+1}^{r_1-1} g(x_1) \left\{ n_2 \delta[p(x_1)] + (N - n_1 - n_2) \sum_{x_2=0}^{a_2-x_1} g(x_2|x_1) [p(x_1, x_2) - pr] \right\}
\]

Then using;

\[
\Delta R_2 = \frac{\Delta k_2}{A_2 - R_2}
\]

\[
= \sum_{x_1=a_1+1}^{r_1-1} g(x_1) \left\{ n_2 \delta[p(x_1)] + (N - n_1 - n_2) \sum_{x_2=0}^{a_2-x_1} g(x_2|x_1) [p(x_1, x_2) - pr] \right\}
\]

\[
- \sum_{x_1=a_1+1}^{r_1-1} g(x_1) \left\{ n_2 \delta[p(x_1)] + (N - n_1 - n_2) \sum_{x_2=0}^{a_2-x_1} g(x_2|x_1) [p(x_1, x_2) - pr] \right\}
\]

... (20)

We can find \([\Delta r_1, R_2(r_1)]\) as;

\[
\Delta R_2(r_1) = g_{n_1}(r_1) \left\{ n_2 \delta[p_{n_1}^{(r_1)}] + (N - n_1 - n_2) \sum_{x_2=0}^{a_2-r_1} g_{n_2}(x_2|x_1) [p_{n_1+n_2}^{(r_1, x_2)} - pr] \right\}
\]

And also;

\[
\Delta R_2(a_1) = \sum_{x_1=a_1+2}^{r_1-1} g(x_1) \left\{ n_2 \delta[p(x_1)] + (N - n_1 - n_2) \sum_{x_2=0}^{a_2-x_1} g(x_2|x_1) [p(x_1, x_2) - pr] \right\}
\]

\[
- \sum_{x_1=a_1+2}^{r_1-1} g(x_1) \left\{ n_2 \delta[p(x_1)] + (N - n_1 - n_2) \sum_{x_2=0}^{a_2-x_1} g(x_2|x_1) [p(x_1, x_2) - pr] \right\}
\]

... (22)

And also the difference of \([R_2(a_1)]\) with respect to \((a_1)\):

\[
\Delta_n_1 R_2(a_1) = g_{n_1}(a_1 + 1) \left\{ n_2(A_2 - S_1) \left[ p_{n_1}^{(a_1+1)} \right] - (S_1 - A_1)(A_2 - R_2) + g_{n_1}(a_1 + 1)(N - n_1 - n_2) \sum_{x_2=a_2-a_1}^{n_2} g_{n_2}(x_2|a_1+1) \left[ p_{n_1+n_2}^{(a_1+1, x_2)} - pr \right] \right\}
\]

... (23)

From \([\Delta r_2, R_2(a_2)]\) we can find the value of parameter \((a_2)\) for given \((n_1, n_2)\) which satisfy the inequality (24) follows.

\[
\hat{\theta}_{n_1+n_2}(\hat{a}_2) \leq pr \leq \hat{\theta}_{n_1+n_2}(\hat{a}_2 + 1)
\]

... (24)

Then;

\[
\hat{r}_1 = \hat{a}_2 + 1 \text{ when } ks = kr
\]

\[
\hat{r}_1 \leq \hat{a}_2 + 1 \text{ for } ks > kr
\]
And when \([a_2 = a_1]\), the second term of \(\Delta_{a_1} R_2(a_1)\) is positive, so we assume that \([a_1 = a'_1]\) is the value at which the first term of \(\Delta_{a_2} R_2(a_2)\) is positive therefore \(\Delta R_2(a'_1) > 0\) and when \(ks = kr\) \(\Rightarrow (\hat{r}_1 = \hat{a}_2 + 1)\), and we see that the average cost of inspection equal to average cost of reject, therefore we need only to find the parameters \((a_1, a_2)\) and from \(\Delta_{a_1} R_2(a_1)\) we obtain \(p_{n_1}(\hat{a}_1 + 1) \leq pr\) subject to \(r_1 \leq n_1 + 1\).

After completing model for double Bayesian sampling inspection plan, this model is applied to distribution of Beta – Binomial.

From \(\Delta_{a_2} R_2(a_2)\) we can find \((a_2)\) that satisfy the inequality (24).

\[
p_{n_1+n_2}(\hat{a}_2) \leq pr \leq p_{n_1+n_2}(\hat{a}_2 + 1) \quad \text{................................. (25)}
\]

When \((a_1 = a_2)\), the second term of \(\Delta_{a_1} R_2(a_1)\) is positive, so we assume that \((a_1 = a'_1)\) is the value of which the first term of \(\Delta_{a_2} R_2(a_2)\) is positive therefore \(\Delta R_2(a'_1) > 0\).

When;

\[
ks = kr
\]

Then;

\[
\Delta R_2(r_1) \leq 0 \Rightarrow \hat{r}_1 = \hat{a}_2 + 1
\]

And for \(ks > kr\) \(\Rightarrow \hat{r}_1 \leq \hat{a}_2 + 1\)

And when \(ks = kr\), we saw that average cost of inspection equal average cost of reject, therefore we need only to find the parameters \((a_1, a_2)\), and from equation \(\Delta_{a_1} R_2(a_1)\), we obtain \(p_{n_1}(\hat{a}_1 + 1) \leq pr\) subject to \(r_1 \leq n_1 + 1\).

4 - APPLICATION OF PROPOSED MODEL

After the total cost function are built, it is necessary to apply this model for sampling distribution to show how the parameters of double sampling plans are obtained, to take a decision for accept or reject, assume \((x_1, x_2, x_3)\) are three independent random variables follow Binomial distribution as:

\[
x_1 \sim b(n_1, p)
\]

\[
x_2 \sim b(n_2, p)
\]

\[
x_3 \sim b(n_3, p)
\]

And \((p)\) is random variable varied from lot to lot, and it's prior distribution \([f(p)]\) is assumed here Beta \((s, t)\), (Beta prior), so the joint probability distribution of \((x_1, x_2, x_3)\) is:

\[
p(x_1, x_2, x_3) = \int_0^1 C_{x_1}^{n_1} p^{x_1} q^{n_1-x_1} C_{x_2}^{n_2} p^{x_2} q^{n_2-x_2} C_{x_3}^{n_3} p^{x_3} q^{n_3-x_3} \frac{1}{\text{Beta}(s, t)} p^{s-1} q^{t-1} dp
\]

\[
= \frac{C_{x_1}^{n_1} C_{x_2}^{n_2} C_{x_3}^{n_3}}{\text{Beta}(s, t)} \int_0^1 p^{x_1+x_2+x_3+s-1} q^{n_1+n_2+n_3-x_1-x_2-x_3+t-1} dp
\]

\[
= \frac{C_{x_1}^{n_1} C_{x_2}^{n_2} C_{x_3}^{n_3}}{\text{Beta}(s, t)} \text{Beta}(x_1 + x_2 + x_3 + s, n_1 + n_2 + n_3 - x_1 - x_2 - x_3 + t) \quad \text{.... (26)}
\]

Also we can find the posterior distribution \(f(p|x_1)\) and \(f(p|x_1, x_2) = f(p|x_1 + x_2)\), and;

\[
p(x_2|x_1) = \int_0^1 b(x_2, n_2, p)f(p|x_1) \, dp
\]

Since;

\[
f(p|x_1) = \frac{\text{joint}(p, x_1)}{g(x_1)}
\]

\[
g(x_1) = \int_0^1 C_{x_1}^{n_1} p^{x_1} q^{n_1-x_1} \frac{1}{\text{Beta}(s, t)} p^{s-1} q^{t-1} dp
\]

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\[ g(x_1) = \frac{C_{x_1}^{n_1}}{\text{Beta}(s, t)} \text{Beta}(x_1 + S, n_1 - x_1 + t) \]

And from equation below which is Beta distribution:
\[ f(p|x_1) = \frac{g(p|x_1)}{g(x_1)} \]

The conditional distribution of \((p)\) given \((x_1)\) is:
\[ f(p|x_1) = \frac{1}{\text{Beta}(x_1+S, n_1-x_1+1)} p^{x_1+S-1} q^{n_1-x_1+t-1} \] ............................... (27)

And;
\[ E(p|x_1) = \frac{x_1 + S}{n_1 + S + t} \]

Also we can find \( pr(x_2|x_1) \):
\[ pr(x_2|x_1) = \int_0^1 b(x_2, n_2, p) f(p|x_1) \, dp \] ................................. (28)
\[ = \frac{1}{k} \int_0^1 C_{x_2}^{n_2} p^{x_2} q^{n_2-x_2-p^{x_1+S-1} q^{n_1-x_1+t-1}} \, dp \]

\[ pr(x_2|x_1) = \frac{1}{k} C_{x_2}^{n_2} \text{Beta}(x_1 + x_2 + S, n_1 + n_2 - x_1 - x_2 + t) \]

\[ pr(x_2|x_1) = \frac{s+x_1}{s+x_1+x_2} \text{Beta}(x_1 + x_2 + S, n_1 + n_2 + x_1 + x_2 + t) \] ................................. (29)

Which is \( [g(x_2|x_1)] \)

Assume \( x_3 = X - x_1 - x_2 \)
\( x_3 \sim \text{Binomial} (n_3, p) \)

Then the conditional distribution of \((x_3)\) given \((x_1, x_2)\) is:
\[ pr(x_3|x_1, x_2) = \int_0^1 b(x_3, n_3, p) f(p|x_1 + x_2) \, dp \] ............................... (30)

To solve equation (30), we need to find \( f(p|x_1 + x_2) \):

Let \( y = x_1 + x_2 \)
\[ \therefore x_1 \sim b(n_1, p) \]
\[ x_2 \sim b(n_2, p) \]

And are independent, then;
\( y = x_1 + x_2 \sim B(n_1 + n_2, p) \)

\[ g(p, y) = C_y^{n_1+n_2} p^y q^{n_1+n_2-y} \frac{1}{B(S, t)} p^{S-1} q^{t-1} \]

\[ \therefore g(y|p) = \frac{1}{B(S, t)} \int_0^1 C_y^{n_1+n_2} p^y q^{n_1+n_2-y} \, dp \]

\[ g(y|p) = \frac{1}{B(S, t)} C_y^{n_1+n_2} \text{Beta}(y+S, n_1+n_2-y+t) \]

\[ \therefore f(p|y = x_1 + x_2) = \frac{\text{joint}}{\text{marginal}} \]
\[ = \frac{1}{B(S, t)} \frac{C_y^{n_1+n_2} p^y q^{n_1+n_2-y} \text{Beta}(y+S, n_1+n_2-y+t)} \]
\[ = \frac{1}{B(S, t)} C_y^{n_1+n_2} \text{Beta}(y+S, n_1+n_2-y+t) \frac{p^y q^{n_1+n_2-y} \text{Beta}(y+S, n_1+n_2-y+t)} \]

\[ 0 \leq p \leq 1 \] ............................... (31)

Then \( f(p|y = x_1 + x_2) \) is distributed as Beta distribution with parameters \((y + S, n_1 + n_2 - y + t)\). This distribution is necessary to find the distribution of number of defectives \((x_3)\) after drawing the first and second sample, i.e to find the distribution of \( p(x_3|x_1, x_2) \).
\[ p(x_3|x_1,x_2) = \int_0^1 b(x_3,n_3,p) f(p|x_1 + x_2 = y) dp \]

\[ = \int_0^1 C_{x_2}^{n_3} p^{x_3} q^{n_3-x_3} \frac{1}{\text{Beta} (y + S, n_1 + n_2 + t - y)} p^{y+S-1} q^{n_1+n_2+t-y-1} dp \]

\[ = \frac{\text{Beta}(y + S, n_1 + n_2 - y + t)}{\text{Beta}(y + S, n_1 + n_2 - y + t)} \frac{\Gamma (x_3 + y + S) \Gamma (n_1 + n_2 + n_3 - x_3 - y + t)}{\Gamma (n_1 + n_2 + t + S) \Gamma (S + t + n_1 + n_2 + n_3)} \]

\[ = \frac{C_{x_3}^{n_3}(n_1+n_2+t+S-1)!}{(y+S-1)(n_1+n_2+t-y-1)!} \frac{(x_3+y+S-1)(n_1+n_2+n_3-x_3-y-t-1)!}{(S+t+n_1+n_2+n_3-1)!} \]

\[ = \frac{C_{x_3}^{n_3} C_{y+S}^{n_1+n_2+t+S-1} (x_3+y+S)!}{C_{x_3+y+S}^{n_3+n_1+n_2+n_3+t+S-1} (y+S)!} \]

Which is mixed hypergeometric distribution. We can obtain the conditional mean of \(x_3\) given \((x_1,x_2)\):

\[ E(x_3|x_1,x_2) = n_3 p(x_1|x_2) \]

Where,

\[ p(x_1,x_2) = p_{n_1+n_2}^{(x_1+x_2)} \]

5- CASE STUDY

The following data represents the percentage of defectives of (140) lots from some industrial companies in Iraq, these percentages are grouped in frequency table as follows:

| Table (1): The frequency table for percentage of defectives of 140 lots |
|---|---|---|---|---|
| Percentage of defectives | Observed frequency \((f_i)\) | Class mark \(p_i\) | \(p_i f_i\) | \(p_i^2 f_i\) |
| 0.0 - 0.02 | 6 | 0.01 | 0.06 | 0.0116 |
| 0.02 - 0.04 | 13 | 0.03 | 0.39 | 0.0117 |
| 0.04 - 0.06 | 17 | 0.05 | 0.85 | 0.0425 |
| 0.06 - 0.08 | 24 | 0.07 | 1.68 | 0.1176 |
| 0.08 - 0.10 | 18 | 0.09 | 1.62 | 0.1456 |
| 0.10 - 0.12 | 16 | 0.11 | 1.76 | 0.1936 |
| 0.12 - 0.14 | 15 | 0.13 | 1.95 | 0.2535 |
| 0.14 - 0.16 | 10 | 0.15 | 1.50 | 0.2250 |
| 0.16 - 0.18 | 9 | 0.17 | 1.53 | 0.2601 |
| 0.18 - 0.20 | 7 | 0.19 | 0.33 | 0.2527 |
| 0.20 - 0.22 | 3 | 0.21 | 0.63 | 0.1323 |
| 0.22 – 0.24 | 2 | 0.23 | 0.46 | 0.1058 |
| Total | 140 | | 13.699 | 1.7413 |
Then we draw the histogram and frequency curve, the curve indicate it is Beta distribution with parameters (S, T), which are estimated by moments method using the equations:

\[
\hat{S} = \frac{\bar{x}_p \bar{x}_q - S^2_p}{\bar{x}_p} \quad \hat{T} = \bar{x}_q \left[ \frac{\bar{x}_p \bar{x}_q - S^2_p}{\bar{x}_p} \right] S^2_p
\]

Where:

\[
\bar{x}_p = \frac{\sum p f_i}{\sum f_i} = 0.09785
\]

\[
S^2_p = \frac{\sum p_i f_i}{\sum f_i} - \bar{p}^2 = 0.00286
\]

Then;

\[
\hat{s}_{mom} = 2.9 \approx 3 \quad \hat{t}_{mom} = 26.9 \approx 27
\]

Then the distribution of \((p_i)\) is tested using \((\chi^2)\) test, where the cumulative probabilities are computed from:

\[
F(p) = \int_0^p \frac{1}{B(S,T)} u^{S-1} (1-u)^{T-1} du
\]

\[
= E(S,S + T - 1, p) = pr(x > S) = \sum_{x=S}^{S+T-1} c_{x}^{a} S^{a} x^{b} q^{S+T-x-1}
\]

Where:

\[
\hat{s} = 3 \quad \hat{t} = 27
\]

The hypothesis tested is;

\(H_0: p \sim Beta(S,T)\)

\(H_1: p \not\sim Beta(S,T)\)

Using Chi – square test, which depend on the following formula;

\[
\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}
\]

The following table represents the necessary computations for goodness of fit.

<table>
<thead>
<tr>
<th>Proportion of defectives</th>
<th>Observed frequency ((O_i))</th>
<th>(pr(p_1 &lt; p &lt; p_2))</th>
<th>(E_i)</th>
<th>(\frac{(O_i - E_i)^2}{E_i})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0 -</td>
<td>6</td>
<td>0.0231</td>
<td>3.234</td>
<td>2.3657</td>
</tr>
<tr>
<td>0.02 -</td>
<td>13</td>
<td>0.0974</td>
<td>13.636</td>
<td>0.0962</td>
</tr>
<tr>
<td>0.04 -</td>
<td>17</td>
<td>0.1662</td>
<td>23.268</td>
<td>1.6384</td>
</tr>
<tr>
<td>0.06 -</td>
<td>24</td>
<td>0.1612</td>
<td>22.568</td>
<td>0.0908</td>
</tr>
<tr>
<td>0.08 -</td>
<td>18</td>
<td>0.1500</td>
<td>21.00</td>
<td>0.4285</td>
</tr>
<tr>
<td>0.10 -</td>
<td>16</td>
<td>0.1252</td>
<td>17.528</td>
<td>0.1332</td>
</tr>
<tr>
<td>0.12 -</td>
<td>15</td>
<td>0.0827</td>
<td>11.578</td>
<td>1.0114</td>
</tr>
<tr>
<td>0.14 -</td>
<td>10</td>
<td>0.0715</td>
<td>10.01</td>
<td>0.00001</td>
</tr>
<tr>
<td>0.16 -</td>
<td>9</td>
<td>0.0435</td>
<td>6.09</td>
<td>1.2904</td>
</tr>
<tr>
<td>0.18 -</td>
<td>7</td>
<td>0.0373</td>
<td>5.222</td>
<td>0.0605</td>
</tr>
<tr>
<td>0.20 -</td>
<td>3</td>
<td>0.0252</td>
<td>3.528</td>
<td>0.0790</td>
</tr>
<tr>
<td>0.22 – 0.24</td>
<td>2</td>
<td>0.0167</td>
<td>2.338</td>
<td>0.0488</td>
</tr>
<tr>
<td>Total</td>
<td>140</td>
<td>1</td>
<td>140</td>
<td>7.1931</td>
</tr>
</tbody>
</table>
Comparing \( \chi^2_{calc} = 7.1931 \) with \( \chi^2_{tab(0.05,8)} = 15.507 \), we find that, \( \chi^2_{calc} < \chi^2_{tab} \), we accept the hypothesis that, \( p_i \sim Beta \) distribution with parameters (S,T). Then the prior distribution of percentage of defective is Beta with \( \hat{S} = 3 \), \( \hat{T} = 27 \).

After we know that the prior distribution of percentage of defective \( f(p) \) is Beta - prior with the estimated parameters \( (\hat{S}, \hat{T}) \) we also obtain the parameters of quality control cost for each produced units, which are;

\[
\begin{align*}
A_1 &= 0, R_1 = 10.000 \text{ ID} \\
S_1 &= 10,000 \text{ ID} \\
R_2 &= 28,000 \text{ ID} \\
S_2 &= 28,000 \text{ ID} \\
A_2 &= 153,000 \text{ ID} \\
pr &= \frac{R_1 - A_1}{A_2 - R_2} = 0.08, \text{ which is the break even quality control point.}
\end{align*}
\]

According to the above parameters we find the parameters of single Bayesian sampling inspection plan from;

\[
\begin{align*}
n^*_1 &= \lambda_1 \sqrt{N} + \lambda_2 \\
C^*_1 &= n^*_1 pr + B_1 \\
B_1 &= \hat{T}pr - \hat{S}qr - \frac{1}{2} \\
p &= \frac{\bar{Y}}{pr} = 1.2 \\
pr &= 0.08 \\
\text{And;} \\
\lambda^2_1 &= \frac{p\hat{S} q^T(\lambda_2 - R_2)}{2B(S,T)(ks-km)} \\
ks &= S_1 + S_2\bar{p} \\
kkm &= kr - (A_2 - R_2) \int_0^{pr} (pr - p)f(p)dp \\
\lambda_2 &= \text{is a constant obtained according to the prior distribution of percentage of defective as;} \\
\lambda_2 &= [3(S + T)^2 - 11(S + T) - 2 - (3S - 1)Spr - (3T - 1)T|qr - 1|prqr]/12 \\
\text{And} \\
kkm &= kr - (A_2 - R_2) \int_0^{pr} (pr - p)f(p)dp
\end{align*}
\]

Since \( \hat{S} = 3 \), \( \hat{T} = 27 \), \( \lambda_1 = 4.663 \), \( \lambda_2 = -27 \), \( N = 200 \)

The parameters of single Bayesian sampling inspection plan are;

\[
\begin{align*}
n^* &= \lambda_1 \sqrt{N} + \lambda_2 = 39 \\
C^* &= n^* pr + B_1 = 2
\end{align*}
\]

\( \therefore \) parameters of single Bayesian sampling plan is; \( (n^* = 39, C^* = 2) \), for different values of \( \bar{Y} = 0.07(0.05)0.09 \) for \( N = 200, 225, 250, 275, 300 \), the following table summarize single sampling plans.
From the data we find the mean of percentage of defective is \( \bar{p} = 0.09785 \). The single Bayesian sampling inspection plan is found \( (n_1^* = 39, C_1^* = 2) \), which defined as drawing at random from lot \( (N = 200) \), a sample of size \( (n_1^* = 39) \), and testing it's units, if the number of defective is found \( (C_1^* = 2) \), then accept from first sample \( (x_1 \leq 2) \) but if \( (x_1 \geq 2) \) then reject from first sample and also stop, but according to the model, we find the single Bayesian sampling inspection plans \( (n_1, C_1, R_1) \) as tabulated in table (5), then the second Bayesian sampling inspection plans are found and put in table (4).

<table>
<thead>
<tr>
<th>( N )</th>
<th>( n_1 )</th>
<th>( c_1 )</th>
<th>( R_1 )</th>
<th>( n_1 )</th>
<th>( c_1 )</th>
<th>( R_1 )</th>
<th>( n_1 )</th>
<th>( c_1 )</th>
<th>( R_1 )</th>
<th>( n_1 )</th>
<th>( c_1 )</th>
<th>( R_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>150</td>
<td>10</td>
<td>0</td>
<td>71.6</td>
<td>15</td>
<td>0</td>
<td>80.16</td>
<td>25</td>
<td>1</td>
<td>102.4</td>
<td>17</td>
<td>0</td>
<td>70.9</td>
</tr>
<tr>
<td>175</td>
<td>13</td>
<td>0</td>
<td>81.8</td>
<td>19</td>
<td>1</td>
<td>94.37</td>
<td>27</td>
<td>1</td>
<td>116.0</td>
<td>22</td>
<td>1</td>
<td>87.9</td>
</tr>
<tr>
<td>200</td>
<td>17</td>
<td>0</td>
<td>95.4</td>
<td>23</td>
<td>1</td>
<td>107.9</td>
<td>30</td>
<td>2</td>
<td>126.2</td>
<td>27</td>
<td>1</td>
<td>104.9</td>
</tr>
<tr>
<td>225</td>
<td>20</td>
<td>0</td>
<td>105.65</td>
<td>26</td>
<td>1</td>
<td>118.52</td>
<td>35</td>
<td>2</td>
<td>143.2</td>
<td>32</td>
<td>2</td>
<td>120.3</td>
</tr>
<tr>
<td>250</td>
<td>23</td>
<td>1</td>
<td>115.82</td>
<td>29</td>
<td>1</td>
<td>128.4</td>
<td>38</td>
<td>2</td>
<td>153.6</td>
<td>36</td>
<td>2</td>
<td>135.6</td>
</tr>
<tr>
<td>275</td>
<td>26</td>
<td>1</td>
<td>126.02</td>
<td>32</td>
<td>2</td>
<td>138.6</td>
<td>41</td>
<td>3</td>
<td>163.6</td>
<td>40</td>
<td>2</td>
<td>149.2</td>
</tr>
<tr>
<td>300</td>
<td>29</td>
<td>1</td>
<td>135.22</td>
<td>35</td>
<td>2</td>
<td>148.8</td>
<td>45</td>
<td>3</td>
<td>177.2</td>
<td>43</td>
<td>2</td>
<td>159.4</td>
</tr>
</tbody>
</table>

After we find the single Bayesian sampling inspection plans, from minimizing \( (R_1) \), then minimizing \( (R_2) \) gives the double Bayesian sampling inspection plans, which works on minimizing total expected quality control cost, derived in equation (10) and then transformed to standard cost function \( (R_2) \) in equation (14).

<table>
<thead>
<tr>
<th>( N )</th>
<th>( n_2 )</th>
<th>( a_2 )</th>
<th>( R_2/R_1 )</th>
<th>( n_2 )</th>
<th>( a_2 )</th>
<th>( R_2/R_1 )</th>
<th>( n_2 )</th>
<th>( a_2 )</th>
<th>( R_2/R_1 )</th>
<th>( n_2 )</th>
<th>( a_2 )</th>
<th>( R_2/R_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>150</td>
<td>7</td>
<td>1</td>
<td>1.00</td>
<td>17</td>
<td>1</td>
<td>0.996</td>
<td>20</td>
<td>2</td>
<td>1.008</td>
<td>18</td>
<td>2</td>
<td>0.996</td>
</tr>
<tr>
<td>175</td>
<td>13</td>
<td>2</td>
<td>0.9</td>
<td>21</td>
<td>2</td>
<td>0.986</td>
<td>25</td>
<td>2</td>
<td>1.06</td>
<td>21</td>
<td>2</td>
<td>0.986</td>
</tr>
<tr>
<td>200</td>
<td>16</td>
<td>1</td>
<td>0.88</td>
<td>23</td>
<td>2</td>
<td>0.97</td>
<td>28</td>
<td>2</td>
<td>0.99</td>
<td>26</td>
<td>2</td>
<td>0.977</td>
</tr>
<tr>
<td>225</td>
<td>19</td>
<td>2</td>
<td>0.85</td>
<td>25</td>
<td>2</td>
<td>0.85</td>
<td>32</td>
<td>2</td>
<td>0.97</td>
<td>30</td>
<td>3</td>
<td>0.967</td>
</tr>
<tr>
<td>250</td>
<td>20</td>
<td>2</td>
<td>0.80</td>
<td>29</td>
<td>2</td>
<td>0.83</td>
<td>36</td>
<td>2</td>
<td>0.88</td>
<td>32</td>
<td>3</td>
<td>0.933</td>
</tr>
<tr>
<td>275</td>
<td>23</td>
<td>1</td>
<td>0.75</td>
<td>31</td>
<td>3</td>
<td>0.79</td>
<td>40</td>
<td>3</td>
<td>0.86</td>
<td>36</td>
<td>3</td>
<td>0.942</td>
</tr>
<tr>
<td>300</td>
<td>30</td>
<td>1</td>
<td>0.74</td>
<td>36</td>
<td>3</td>
<td>0.75</td>
<td>42</td>
<td>3</td>
<td>0.85</td>
<td>38</td>
<td>4</td>
<td>0.931</td>
</tr>
</tbody>
</table>

From the data we find the mean of percentage of defective is \( \bar{p} = 0.09785 \). The single Bayesian sampling inspection plan is found \( (n_1^* = 39, C_1^* = 2) \), which defined as drawing at random from lot \( (N = 200) \), a sample of size \( (n_1^* = 39) \), and testing it's units, if the number of defective is found \( (C_1^* = 2) \), then accept from first sample \( (x_1 \leq 2) \) but if \( (x_1 \geq 2) \) then reject from first sample and also stop, but according to the model, we find the single Bayesian sampling inspection plans \( (n_1, C_1, R_1) \) as tabulated in table (5), then the second Bayesian sampling inspection plans are found and put in table (4).

**Table (4): Double Bayesian sampling inspection plans**

<table>
<thead>
<tr>
<th>( N )</th>
<th>( n_2 )</th>
<th>( a_2 )</th>
<th>( R_2/R_1 )</th>
<th>( n_2 )</th>
<th>( a_2 )</th>
<th>( R_2/R_1 )</th>
<th>( n_2 )</th>
<th>( a_2 )</th>
<th>( R_2/R_1 )</th>
<th>( n_2 )</th>
<th>( a_2 )</th>
<th>( R_2/R_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>150</td>
<td>7</td>
<td>1</td>
<td>1.00</td>
<td>17</td>
<td>1</td>
<td>0.996</td>
<td>20</td>
<td>2</td>
<td>1.008</td>
<td>18</td>
<td>2</td>
<td>0.996</td>
</tr>
<tr>
<td>175</td>
<td>13</td>
<td>2</td>
<td>0.9</td>
<td>21</td>
<td>2</td>
<td>0.986</td>
<td>25</td>
<td>2</td>
<td>1.06</td>
<td>21</td>
<td>2</td>
<td>0.986</td>
</tr>
<tr>
<td>200</td>
<td>16</td>
<td>1</td>
<td>0.88</td>
<td>23</td>
<td>2</td>
<td>0.97</td>
<td>28</td>
<td>2</td>
<td>0.99</td>
<td>26</td>
<td>2</td>
<td>0.977</td>
</tr>
<tr>
<td>225</td>
<td>19</td>
<td>2</td>
<td>0.85</td>
<td>25</td>
<td>2</td>
<td>0.85</td>
<td>32</td>
<td>2</td>
<td>0.97</td>
<td>30</td>
<td>3</td>
<td>0.967</td>
</tr>
<tr>
<td>250</td>
<td>20</td>
<td>2</td>
<td>0.80</td>
<td>29</td>
<td>2</td>
<td>0.83</td>
<td>36</td>
<td>2</td>
<td>0.88</td>
<td>32</td>
<td>3</td>
<td>0.933</td>
</tr>
<tr>
<td>275</td>
<td>23</td>
<td>1</td>
<td>0.75</td>
<td>31</td>
<td>3</td>
<td>0.79</td>
<td>40</td>
<td>3</td>
<td>0.86</td>
<td>36</td>
<td>3</td>
<td>0.942</td>
</tr>
<tr>
<td>300</td>
<td>30</td>
<td>1</td>
<td>0.74</td>
<td>36</td>
<td>3</td>
<td>0.75</td>
<td>42</td>
<td>3</td>
<td>0.85</td>
<td>38</td>
<td>4</td>
<td>0.931</td>
</tr>
</tbody>
</table>

**CONCLUSION**

1- Finding plan which minimize the linear combination of two expected cost function for first sampling and second sampling required a procedure carried out by means of difference equation, since the cost function is of discrete type, and the minimization is done in two stages to find \( (R_1) \) and \( (R_2) \).

2- We try also to find a family of optimum plans through maximizing the OC curve, but it depend on probability of sampling under double sampling plans, and do not consider the parameters of cost function.

3- The model was built under Beta – Binomial distribution since the distribution of process is Binomial \( (n, p) \), and \( (p) \) is random variable have \( f(p) \), and it found to be Beta \( (S, T) \).

4- The parameters of \( f(p) \), \( (S, T) \) are estimated by moments method and it can be estimated also by any other statistical method.
5- The decision to accept or reject obtained from second sample is more effective than the decision just obtained from first sampling especially for expensive units.
6- The producer risk for the studied company was \((B = 0.10)\), but it is found greater than this number, because of bad station of production in Iraq, but in spite of this the company work to satisfy the required quality control level.
7- The percentage of defectives in 140 lot are varied from to lot, so the distribution of these percentage is considered, and it is found to be Beta prior with estimated parameters \((\hat{S}, \hat{\theta})\), therefore the model is build for Bayesian sampling plan.

REFERENCES