BUILDING COMPOSITE EXPONENTIAL – BURR TYPE XII DISTRIBUTION

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ABSTRACT

This paper deals with constructing a composite probability distribution, which can be used for representing the failure time distribution of mixed devices, and mixed component. This distribution is called a composite exponential Burr – type XII model, it consist of exponential density up to certain value of threshold parameter, the second is a four parameter Burr – type XII for the rest of model, this give better fitting for data rather than single model like exponential alone or single Burr – type XII. The p.d.f of this distribution has five parameters (θ, λ, β, σ, μ), according to certain imposed conditions on the function and on its derivatives at the threshold parameter (μ), this help us to find some mathematical relations between parameters, therefore the number of five unknown parameters is reduced to two unknown parameters rather than five.

In this paper the probability density of this composite distribution is derived, also its cumulative distribution function (C.D.F) also obtained. Finally the parameters (β, σ) are estimated using maximum likelihood method for ordered observations.

Keywords: Composite Distribution, Exponential Burr – Type XII, Threshold & Location Parameter (μ), Shape Parameters (λ, β), Scale Parameters (σ, θ).

1. INTRODUCTION

Sometimes the data of failure rate, or insurance claims, or the data represents some effect, are skewed heavy tailed, it cannot represented by simple parametric probability model, so we work here to compose a probability distribution from mixing an exponential distribution when its values between zero and the value of threshold parameter (μ), the second distribution is Burr – type XII with four parameters (Zimmer etal 1998), the composite distribution is appropriate model for analyzing data of failure time distribution of mixed devices. Also it can be used for data with heavy
tailed, and also for data have two shaped parameters, and two scaled parameters. The composite distribution and also the generalized distribution have many applications to analysis failure time data and survival data (Nelson 1982, Ortega Cancho 2006), many research conducted to evaluate the performance of composite distribution with log Burr – type XII, and log – logistic model, which are necessary in survival analysis (Lawless, J.F.,2003). These composite models are necessary to model, the lifetime when the failure rate represented by some function for the next value of R.V.X. The first section of this article contains the definition of composite distribution, and how to use some imposed conditions to find mathematical relation between parameters and to simplify the operation of estimating parameters. In section 2 we consider the estimate of two parameters (β, σ) using maximum likelihood of ordered observations.

2. DEFINITION OF COMPOSITE DISTRIBUTION

Let (X) be random variable have;

\[ f_1(x) = \theta e^{-\beta x} \quad x > 0 \quad (1) \]

\[ f_2(x) = \frac{\lambda \beta^\lambda}{\sigma} \left[ \beta + \left(\frac{x-\mu}{\sigma}\right)\right]^{-\lambda-1} \quad x > \mu \quad \lambda > 0 \quad \beta > 0 \quad (2) \]

\[ -\infty < \mu < \infty \]

0 \quad o/w

Be a four parameter Burr XII distribution, then the composite function;

\[ f(x) = \begin{cases} 
  c f_1(x) & 0 < x \leq \mu \\
  c f_2(x) & \mu < x \leq \infty 
\end{cases} \quad (3) \]

To obtain the composite p.d.f and then, working on estimating it’s parameters, we can introduce some conditions from continuity and differentiability, like first and second derivative, and solving these condition at the threshold of distribution (μ), this help us to find some mathematical relations between parameters and then to reduce the number of parameters for composite density, to simplify method of estimating parameters.

But here we apply the conditions;

\[ f_1(\mu) = f_2(\mu) \quad (4) \]

Then from first condition;

\[ f_1(\mu) = f_2(\mu) \]

We have:

\[ \mu e^{-\mu^2} = \frac{\lambda \beta^\lambda}{\sigma} \beta^{-\lambda-1} \]

\[ \mu e^{-\mu^2} = \frac{\lambda}{\beta \sigma} \quad \text{first condition} \]

And from second condition;

\[ f_1(\mu) = f_2(\mu) \]

Were \( f_1'(x) = -\mu^2 e^{-\mu x} \) and;

\[ f_2'(x) = \frac{\lambda \beta^\lambda}{\sigma} (-\lambda - 1) \left[ \beta + \left(\frac{x-\mu}{\sigma}\right)\right]^{-\lambda-2} \frac{1}{\sigma} \]
\[-\frac{\lambda (\lambda + 1)}{\sigma^2} \beta \lambda \left[ \beta + \left( \frac{\chi - \mu}{\sigma} \right) \right]^{-\lambda - 2} \]

\[f'_2(\mu) = \frac{-\lambda (\lambda + 1)}{\sigma^2 \beta^2} \]

\[f'_1(\mu) = -\mu^2 e^{-\mu^2} \]

The second condition reduced to:

\[\frac{\lambda (\lambda + 1)}{\sigma^2 \beta^2} = \mu^2 e^{-\mu^2} \]

\[\frac{\lambda (\lambda + 1)}{\sigma^2 \beta^2} = \theta^2 e^{-\theta \mu} \]

After simplification, the result is:

\[\theta^2 e^{-\theta \mu} \left( 1 + \frac{1}{\lambda} - e^{\theta \mu} \right) = 0 \]

\[e^{\theta \mu} = 1 + \frac{1}{\lambda} \]

\[\theta \mu = \ln \left( 1 + \frac{1}{\lambda} \right) \]

\[\lambda \left( 1 + \frac{1}{\lambda} \right) = \beta \sigma \theta \]

\[\lambda = \beta \sigma \theta - 1 \]

For each value of (\(\lambda\)), (i.e), \(\lambda = \frac{1}{2}\)

\[\beta \sigma \theta = 1.5 \]

And for \(\lambda = 2\)

\[\beta \sigma \theta = 3\] and so on.

Since:

\[\theta = \frac{\lambda + 1}{\beta \sigma} \]

And \(\theta \mu = \ln \left( 1 + \frac{1}{\lambda} \right)\)

\[\theta = \frac{\ln \left( 1 + \frac{1}{\lambda} \right)}{\mu} \]

\[\frac{\lambda + 1}{\beta \sigma} = \frac{\ln \left( 1 + \frac{1}{\lambda} \right)}{\mu} \]

When the shape parameter (\(\lambda\)), of Burr type XII is known, and also the threshold parameter (\(\mu\)) is known we can estimate (\(\beta, \sigma\)) by using maximum likelihood for ordered observation, then (\(\hat{\beta}_{MLE}, \hat{\sigma}_{MLE}\)) can be used to find (\(\hat{\mu}_{MLE}\)) (because of the invariant property of maximum likelihood estimator), but before this, the value of constant (c) must be found;

Applying the condition:

\[\int_0^\infty f(x)dx = 1 \]

\[c \int_0^\infty f(x)dx = 1 \]

\[c \left[ \int_0^\infty f_2(x)dx + \int_0^\infty f_1(x)dx \right] = 1 \]
Therefore the composite density from exponential and Burr – type XII becomes:

\[
f_X(x) = \begin{cases} 
0.564436 \theta e^{-\theta x} & 0 < x \leq \mu \\
0.564436 \frac{\lambda \beta^\lambda}{\sigma} \left[ \beta + \left( \frac{x - \mu}{\sigma} \right) \right]^{-\lambda - 1} & x > \mu
\end{cases}
\]  
\hspace{1cm} (5)

since \( \theta = \frac{1.5}{\beta \sigma} \) therefore;

\[
f_X(x) = \begin{cases} 
0.84665 \frac{e^{1.5}}{\beta \sigma} x & 0 < x \leq \mu \\
0.564436 \frac{\lambda \beta^\lambda}{\sigma} \left[ \beta + \left( \frac{x - \mu}{\sigma} \right) \right]^{-\lambda - 1} & x > \mu
\end{cases}
\]  
\hspace{1cm} (6)

While the cumulative distribution function \( (c. d. f) \) of this composite probability distribution is given by:

\[
f_X(x) = \begin{cases} 
0.56443 \left( 1 - e^{-\frac{1.5}{\beta \sigma} x} \right) & 0 \leq x \leq \mu \\
1 - 0.564436 \beta^\lambda \left[ \beta + \left( \frac{x - \mu}{\sigma} \right) \right]^{-\lambda} & \mu \leq x \leq \infty
\end{cases}
\]  
\hspace{1cm} (7)

After the definition of composite distribution, we work now on the estimation of parameters.

3. PARAMETER ESTIMATION

3.1 Percentile Method

The parameter \((\theta = \frac{1.5}{\beta \sigma})\) can be estimated as the \( p^{th} \) percentile where:

\[
p = F(\theta) \]  
\hspace{1cm} (8)
\[ p = 0.56443 \left(1 - e^{-\frac{1.5}{\beta \sigma} \theta}\right) \approx 0.4385 \]

Then for \( p = F(\theta) \) we have:

\[ 0.4385 = 0.56443 \left(1 - e^{-\frac{1.5}{\beta \sigma} \theta}\right) (8^*) \]

Giving initial values for \((\beta, \sigma)\) and solving \((8^*)\) we can find \((\hat{\theta})\) percentile estimator, also from the relation \((\theta = \frac{1.5}{\beta \sigma})\), one can obtain an initial estimator either for \((\hat{\theta})\) or for \((\hat{\sigma})\) numerically.

Now we explain how to obtain maximum likelihood estimator from ordered observation; \([(x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n)]\) taken from composite p.d.f \((6)\).

### 3.2 Maximum Likelihood Method

To obtain the maximum likelihood estimator’s of parameters, of composite distribution, obtained from exponential and Burr type XII (with four parameters), first of all we must arrange the sample observation \((x_1, x_2, x_3, \ldots, x_n)\) as \((x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n)\) be an ordered statistics sample from the composite exponential Burr type XII, we assume the unknown parameter \((\theta = \sigma \beta)\) lies between \(m^{th}\) observations and \((m + 1)^{th}\) observations, i.e;

\[ x_m \leq \theta \leq x_{m+1} \]

\[ \hat{\theta} = (1 - h)x_m + hx_{m+1} \quad (9) \]

With;

\[ m = [(n + 1)p] \]
\[ h = (n + 1)p - m \]

Then \((\hat{\theta})\) is percentile estimator depend on \((p, m, h)\) can be used to estimate \((\sigma)\) or \((\beta)\) from;

\[ \hat{\theta} = \frac{1.5}{\beta \sigma} \quad (10) \]

When \((\hat{\theta})\) computed from \((8)\), it can be used in equation \((10)\) to obtain an initial estimate either of \((\hat{\theta})\) or \((\hat{\sigma})\), this is necessary to simplify the estimator’s obtained from the following (MLE) method. Let \((x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n)\) be an ordered statistics sample from the composite exponential Burr type XII defined in \((6)\). In order to evaluate the likelihood function we must have an idea of where the unknown parameter \((\theta)\), assume it is between \([x_m \text{ and } x_{m+1}, x_m \leq \theta \leq x_{m+1}]\).

The likelihood function is;

\[ L(x_1, x_2, x_3, \ldots, x_n, \theta) = \prod_{i=1}^{m} f(x_i) \prod_{i=m+1}^{n} f(x_i) \]

Therefore;

\[ L(x_1, x_2, x_3, \ldots, x_n, \theta) = \left(\frac{0.84665}{\beta \sigma}\right)^m e^{-\frac{1.5}{\beta \sigma} \sum_{i=1}^{m} x_i} \]

\[ (0.564436)^{n-m} \left(\frac{\lambda \beta \lambda}{\sigma}\right)^{n-m} \prod_{i=m+1}^{n} \left[\beta + \left(\frac{x_i - \mu}{\sigma}\right)\right]^{-\lambda-1} \quad (11) \]
Taking logarithm of \( L \):

\[
\log L = -m \log(\sigma \beta) - \frac{1.5}{\beta \sigma} \sum_{i=1}^{m} x_i + (n - m) \log \left( \frac{\lambda \beta^{1.5}}{\sigma} \right) \\
-(\lambda + 1) \sum_{i=m+1}^{n} \log \left[ \beta + \left( \frac{x_i - \mu}{\sigma} \right) \right] \quad (12)
\]

After simplifying equation (12), we have:

\[
-m \log(\sigma) - m \log(\beta) - 1.5(\beta \sigma)^{-1} \sum_{i=1}^{m} x_i \\
+(n - m)[\log(\lambda) + \lambda \log(\beta) - \log(\sigma)] \\
-(\lambda + 1) \sum_{i=m+1}^{n} \log \left[ \beta + \left( \frac{x_i - \mu}{\sigma} \right) \right] \quad (13)
\]

From deriving equation (13) with respect to \((\beta, \sigma, \lambda)\), we can obtain the maximum likelihood estimator for \((\beta, \sigma, \lambda)\) as:

\[
\log L = -m \log \sigma - m \log \beta - 1.5 \frac{m \bar{x}}{\beta \sigma} \\
+(n - m)[\log(\lambda) + \lambda \log(\beta) - \log(\sigma)] \\
-(\lambda + 1) \sum_{i=m+1}^{n} \log \left[ \beta + \left( \frac{x_i - \mu}{\sigma} \right) \right] \\
\frac{\partial \log L}{\partial \sigma} = -\frac{m}{\sigma} + 1.5 \frac{(\beta)^{-1} \sum_{i=1}^{m} x_i}{\sigma^2} - \frac{(n - m)}{\sigma} \\
-(\lambda + 1) \sum_{i=m+1}^{n} \frac{1}{\beta + \left( \frac{x_i - \mu}{\sigma} \right)} \left[ \frac{x_i}{\sigma^2} \right] = 0 \quad (14)
\]

From \( \frac{\partial \log L}{\partial \sigma} = 0 \)

\[
-m \frac{1}{\sigma} + 1.5 \frac{(\beta)^{-1} \sum_{i=1}^{m} x_i}{\sigma^2} - \frac{(n - m)}{\sigma} \\
-(\lambda + 1) \sum_{i=m+1}^{n} \frac{1}{\beta + \left( \frac{x_i - \mu}{\sigma} \right)} \left[ \frac{x_i}{\sigma^2} \right] = 0
\]

\[
\frac{1.5}{\beta} \sum_{i=1}^{m} x_i + (\lambda + 1) \sum_{i=m+1}^{n} \frac{x_i}{\beta + \left( \frac{x_i - \mu}{\sigma} \right)} = n\sigma
\]

Hence:

\[
\hat{\sigma}_{MLE} = \frac{1}{n} \left\{ \frac{1.5}{\beta} \sum_{i=1}^{m} x_i + (\lambda + 1) \sum_{i=m+1}^{n} \frac{x_i}{\beta + \left( \frac{x_i - \mu}{\sigma} \right)} \right\} \quad (15)
\]

Which is an implicit function of \((\hat{\sigma})\), and \((\hat{\beta})\).
Since we consider \((\lambda)\) known then \((\sigma_{MLE})\) can be obtained from (15), and \((\hat{\beta})\) can be obtained from (16) simultaneously and numerically;

\[
\frac{\partial \log L}{\partial \beta} = -\frac{m}{\beta} + 1.5 \frac{\sum_{i=1}^{m} x_i}{\beta^{2}\sigma^2} - \frac{\lambda(n - m)}{\beta} - (\lambda + 1) \sum_{i=m+1}^{n} \left[ \beta + \left(\frac{x_i - \mu}{\sigma}\right)\right]^{-1}
\]

Putting \(\frac{\partial \log L}{\partial \beta} = 0 \Rightarrow \hat{\beta}_{MLE} = \frac{-1.5 \sum_{i=1}^{m} x_i + (\lambda + 1) \sum_{i=m+1}^{n} \left[ \beta + \left(\frac{x_i - \mu}{\sigma}\right)\right]^{-1}}{\lambda n - m (\lambda + 1)}

(16)

Which is also an implicit function can be solved numerically.

Then from equation (15) and (16) where \((\lambda, \mu)\) are known, \(\sigma_{MLE}, \hat{\beta}_{MLE}\) depend on order observation \(\sum_{i=1}^{m} x_i\) also observations from \((m + 1)\) to \((n)\), for given initial values of \((\beta, \sigma)\), we can solve equation (12) and (13) using any numerical method like Newton Raphson iterative method and simultaneously to obtain \(\sigma_{MLE}, \hat{\beta}_{MLE}\) then since;

\[
\hat{\theta} = \frac{1.5}{\hat{\beta} \hat{\sigma}}
\]

So it is obtained depend on \((\hat{\beta}, \hat{\sigma})\) invariant property for M.L.E.

**CONCLUSION**

1. The composite distribution is necessary for study the data of failure time for system of different independent component.
2. The composite distribution is better fitting model than exponential alone, or Burr type XII alone.
3. The imposed condition on the values of functions and values of derivative of these function, help in reducing the number of five unknown parameters \((\beta, \sigma, \theta, \lambda, \mu)\) to two unknown parameters \((\beta, \sigma)\), which are estimated using maximum likelihood of ordered observation.
4. We can impose another condition using second derivative.

**REFERENCES**


