International Journal of Mechanical Engineering and Technology (IJMET)
Volume 9, Issue 9, September 2018, pp. 1384–1392, Article ID: IJMET_09_09_152
Available online at http://www.iaeme.com/ijmet/issues.asp?JType=IJMET&VType=9&IType=9
ISSN Print: 0976-6340 and ISSN Online: 0976-6359

© **IAEME** Publication **Scopus** Indexed

HIGHER ORDER SUPER-IMPLICIT HYBRID MULTISTEP METHODS FOR SECOND ORDER DIFFERENTIAL EQUATIONS

T.A. Anake*
Department of Mathematics, Covenant University, Ota, Ogun state, Nigeria

I.C. Felix
School of Engineering and Applied Sciences, Lagos City Polytechnic, Ikeja, Nigeria

O.P. Ogundile
Department of Mathematics, Covenant University, Ota, Ogun state, Nigeria
*Corresponding Author Email: timothy.anake@covenantuniversity.edu.ng

ABSTRACT

A generalized set of hybrid parameters suitable for the construction of higher order P- stable super-implicit hybrid linear multistep method is presented in this paper. Based on this generalization, highly stable algorithms for the numerical solutions of special second order initial value problems (IVP) of ordinary differential equations (ODE) are derived. Numerical experiments performed on sample problems revealed that the new method is superior to previous methods.

Key words: P-stability, Super-implicit, Hybrid parameters, Interval of Periodicity.

Cite this Article: T.A. Anake1, I.C. Felix, O.P. Ogundile, Higher Order Super-Implicit Hybrid Multistep Methods for Second Order Differential Equations, *International Journal of Mechanical Engineering and Technology* 9(8), 2018, pp. 1384–1392.
http://www.iaeme.com/IJMET/issues.asp?JType=IJMET&VType=9&IType=8

1. INTRODUCTION

Our task in this paper is to consider the approximate solutions of special second order IVP of the form

$$y''(x) = f(x, y(x)),\ y(x_0) = y_0,\ y'(x_0) = y'_0 \quad x \in [a,b], \tag{1}$$

where the first derivative does not appear explicitly. For second order IVP where the first derivative appears explicitly and their solutions see [1, 2, 3, 13] and the references therein. Equation (1) is known to have inherent "periodic stiffness" [10], which makes it difficult to solve analytically. Numerical methods must be employed to obtain its approximate solution. There is vast literature on approximate solutions of (1), see [4-11] and the references therein. The authors in [4] have clearly shown that the barrier on the attainable order of accuracy of P-stable linear multistep methods (LMM) can be bypassed if certain hybrid methods are considered. Furthermore, improving the accuracy of P-stable methods that are highly efficient for the solution of (1) is often based on: (i) the use of higher derivatives of $f(x,y)$; (ii) incorporating additional stages, off-step points, super-future points and the likes; see [6, 11] and the references therein. The idea of super-implicit methods necessitate the use of both the past, present and future values of the solution values of the ODE, [6]. However, this area of interest has only received limited attention over the years. Recent works on super-implicit formulas can be found in [7], [10], [11] and [12]. In particular, [7] and [11] have independently derived P-stable super-implicit hybrid methods, using a hybrid parameter suggested in [4]. In this present paper, we introduce a new set of hybrid parameters suitable for the construction of higher order P-stable super-implicit hybrid linear multistep methods.

A class of methods for solving (1) numerically is the Linear Multistep Methods (LMM), in which a *k*-step method is described by

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h^2\sum_{j=0}^{k}\beta_j f(x_{n+j}, y_{n+j}),\ \ k\geq 2\ . \tag{2}$$

The method (2) is characterized by the polynomials $\rho(\xi)$ and $\sigma(\xi)$ defined as

$$\rho(\xi) := \sum_{j=0}^{k}\alpha_j\xi^j,\quad \sigma(\xi) := \sum_{j=0}^{k}\alpha_j\xi^j,\quad \xi\in\mathbb{C}, \tag{3}$$

and assumed to satisfy the following hypotheses;

1. $\alpha_j, \beta_j \in \mathbb{R}$ for $j = 0(1)k$ and $\alpha_k = 1$, $|\alpha_0| + |\beta_0| \neq 0;\ \sum_{j=0}^{k}|\beta_j| \neq 0$

2. ρ and σ have no common factor, i.e., $(\rho,\sigma) = 1$

3. The method (2) is consistent if $\rho(1) = \rho'(1),\ \ \rho''(1) = 2\sigma(1)$

4. The method (2) is zero-stable

Definition 1.1 [10]. The order p of the method (2) is defined for an arbitrary test function, $z(x)$ as

$$\sum_{j=0}^{k}\alpha_j z(x+jh) - h^2\sum_{j=0}^{k}\beta_j z''(x+jh) = c_{p+2}h^{(p+2)}(x) + O(h^{(p+3)}), \tag{4}$$

with error constant given by

$$c_q = \frac{1}{q!}\sum_{j=0}^{k} j^{q-2}(j^2\alpha_j - q(q-1)\beta_j) - \sum_{j=0}^{k}\frac{j^{q-2}}{(q-2)!}\beta_j,\quad q\geq 2. \tag{5}$$

Definition 1.2 [10]. The method (2) is said to be convergent if it is consistent and zero-stable.

Definition 1.3 [6]. The method (2) is called symmetric if $\alpha_j = \alpha_{k-j}$ and $\beta_j = \beta_{k-j},\ j = 0,1,\ldots,k.$

Definition 1.4 [10]. The method (2) is said to have interval of periodicity $(0, H^2)$ if for all $H^2 \in (0, H^2)$, the zeros of

$$\Pi(z,H^2)=\rho(z)+H^2\sigma(z)=0, \qquad H=i\lambda h, \tag{6}$$

satisfy;

$$z_1=e^{i\theta(H)}, z_2=e^{-i\theta(H)}, |\, zk\,| \le 1, k \ge 2. \tag{7}$$

Definition 1.5 [10]. The method (2) is said to be P-stable if its interval of periodicity is $(0,\infty)$.

The stability and periodicity interval of numerical schemes for the solution of second order IVP (1) can be examined by applying the numerical method (2) on the scalar test equation;

$$y''+\lambda^2 y=0, \quad \lambda\in\mathbb{R}, \tag{8}$$

which gives the following finite difference equation:

$$\sum_{r=0}^{k} A_r\left(\xi^2\right) y_{n+r}=0, \quad \xi=i\lambda h, \tag{9}$$

where $A_r\left(\xi^2\right)$ are polynomials in ξ^2. The characteristics equation of the general difference equation (9) is described by

$$\Pi\left(R,\xi\right)=\sum_{j=0}^{k} A_r\left(\xi^2\right)R^j, \quad k>1 \tag{10}$$

As in [6], (10) is thus expected to satisfy the following properties

1. $A_0\left(\xi^2\right)\neq 0,$
2. $\Pi(R,\xi)$ is irreducible,
3. $\Pi(1,0)=\dfrac{\partial\Pi}{\partial R}(1,0)=0, and$
4. $\dfrac{\partial^2\Pi}{\partial R^2}(1,0)\neq 0.$

Recall the super-implicit methods developed in [6] namely;

$$\sum_{j=0}^{k} a_j y_{n+1-j}=h^2\sum_{j=0}^{k'} b_j f_{n+1+m-j}. \tag{11}$$

The method is called explicit if $m<0$, implicit if $m=0$ and super-implicit for $m>0$, ([6], [12]).

2. THE PROPOSED METHODS

In this section, we introduce the proposed hybrid parameters and derive a higher order P-stable Super-implicit method. The newly derived method is an extension of the work in [12].

2.1. Formulation of Hybrid Parameters

The new set of hybrid parameters considered here are defined on the interval [0,1] as follows,

$$u_k=\sum_{i=0}^{k}\frac{\lambda}{2^i}, \tag{12}$$

where $\lambda=\frac{1}{2}$ is a particular case suggested in [4] and employed in [7] for the derivation highly stable formulas. The hybrid parameters u_k were tested at every node in the given interval and found to be consistent for the construction of super-implicit hybrid linear multistep methods. P-stable methods obtained based on this considerations have order of accuracy greater than those obtained in [4], [7] and [12]. In some cases the new methods yield order of accuracy as high as the order of the methods in [11]. Therefore, we refer to (12) as generalized hybrid parameters.

2.2. Derivation of Super-Implicit Hybrid Multistep Method

The proposed symmetric super-implicit hybrid linear multistep method considered in this paper is of the form

$$\sum_{j=0}^{\frac{k}{2}}\alpha_j(y_{n+j}+y_{n-j})=h^2\sum_{j=0}^{\frac{s}{2}}\beta_j(f_{n+j}+f_{n-j})+h^2[\beta_v(f_{n+v}+f_{n-v}) + \beta_u(f_{n+v}+f_{n-v})] \quad (13)$$

The method (13) is explicit if $s=k-1$, implicit when $s=k$ and super-implicit for $s>k$ with hybrid parameters $u,v\in[0,1]$. Consider a particular example for which $k=4$ and $s=8$, the coefficients α_j are arbitrarily chosen, say $\alpha_0=\alpha_2=1$ and $\alpha_1=-2$, to satisfy symmetry and zero-stability conditions. So that, $\beta_j,\ j=0(1)\frac{3}{2},\beta_v$ and β_u are then determined. The super-implicit hybrid formula (13) numerically integrates $y_{n\pm v}$ and $y_{n\pm u}$ in terms of the expressions involving the quantities $y_{n\pm j}$ and $y_{n\pm j}$. Hence, the corresponding hybrids for (13) include;

$$y_{n+v}=\sum_{j=0}^{\frac{k}{2}}\alpha_j\left(y_{n+j}+y_{n-j}\right)+h^2\sum_{j=0}^{\frac{s}{2}}\alpha_j\left(f_{n+j}+f_{n-j}\right) \quad (14)$$

$$y_{n-v}=\sum_{j=0}^{\frac{k}{2}}b_j(y_{n+j}+y_{n-j})+h^2\sum_{j=0}^{\frac{s}{2}}\hat{b}_j(f_{n+j}+f_{n-j}) \quad (15)$$

$$y_{n+u}=\sum_{j=0}^{\frac{k}{2}}c_j(y_{n+j}+y_{n-j})+h^2\sum_{j=0}^{\frac{s}{2}}\hat{c}_j(f_{n+j}+f_{n-j}) \quad (16)$$

$$y_{n-u}=\sum_{j=0}^{\frac{k}{2}}d_j(y_{n+j}+y_{n-j})+h^2\sum_{j=0}^{\frac{s}{2}}\hat{d}_j(f_{n+j}+f_{n-j}) \quad (17)$$

where k and s are positive even integers and u,v are obtained from (12) to achieve P-stability. The application of (13) on the scalar test equation (8) will result to

$$\sum_{j=0}^{\frac{k}{2}}\alpha_j(\alpha_j+h^2\lambda^2\beta_j)+(y_{n+j}+y_{n-j})+h^2\sum_{j=0}^{\frac{s}{2}}\beta_j(y_{n+j}+y_{n-j}) + h^2\lambda^2[\beta_v(y_{n+v}+y_{n-v})+\beta_u(y_{n+u}+y_{n-u})]=0 \quad (18)$$

Thus, if we substitute $y_n = e^{i\lambda x_n}$ and $z = \lambda h$ in (18), we have

$$\sum_{j=0}^{\frac{k}{2}} (\alpha_j + z^2\beta_j)\cos(jz) + \sum_{j=\frac{k}{2}+1}^{\frac{s}{2}} z^2\beta_j \cos(jz) + z^2[\beta_v \cos(vz) + \beta_v \cos(uz)] = 0 \quad (19)$$

Now for $k = 4$ and $s = 8$, the method (13) becomes

$$y_{n+2} - 2y_{n+1} + 2y_n - 2y_{n-1} - y_{n-1} + y_{n-2} = \quad (20)$$
$$h^2[\beta_4(f_{n+4} + f_{n-4}) + \beta_3(f_{n+3} + f_{n-3}) + \beta_2(f_{n+2} + f_{n-2}) + \beta_1(f_{n+1} + f_{n-1})$$
$$+ 2\beta_0 f_n] + h^2[\beta_v(f_{n+v} + f_{n-v}) + \beta_u(f_{n+u} + f_{n-u})].$$

Given the choices $v = \frac{3}{4}$ and $u = \frac{5}{8}$, we obtained the following coefficients;

$$\beta_0 = \frac{5125533389}{3891888000},\ \beta_1 = \frac{21551964191}{5902696800},\ \beta_2 = \frac{3801449647}{9156746000},\ \beta_3 = -\frac{4073330777}{11258259012000}, \quad (21)$$

$$\beta_4 = -\frac{4641323}{714762921600},\ \beta_u = -\frac{223234121739010048}{20476310398669125},\ \beta_v = -\frac{12331239866368}{1004380752375},$$

and principal local truncation error as $-\frac{11153219171}{114891223832985600} y^{(16)}(x)h^{16} + O\left(h^{17}\right)$.

Similarly, for $k = 4$ and $s = 8$, the hybrids formulae are obtained as

$$y_{n+v} = a_2 y_{n+1} + a_1 y_n + a_0 y_{n-1} + h^2[\hat{a}_4 f_{n+4} + \hat{a}_3 f_{n+3} + \hat{a}_2 f_{n+2} + \hat{a}_1 f_{n+1} + \hat{a}_0 f_n] \quad (22)$$

$$y_{n-u} = b_2 y_{n+1} + b_1 y_n + b_0 y_{n-1} + h^2[\hat{b}_4 f_{n+4} + \hat{b}_3 f_{n+3} + \hat{b}_2 f_{n+2} + \hat{b}_1 f_{n+1} + \hat{b}_0 f_n] \quad (23)$$

$$y_{n+u} = c_2 y_{n+1} + c_1 y_n + c_0 y_{n-1} + h^2[\hat{c}_4 f_{n+4} + \hat{c}_3 f_{n+3} + \hat{c}_2 f_{n+2} + \hat{c}_1 f_{n+1} + \hat{c}_0 f_n] \quad (24)$$

$$y_{v-u} = d_2 y_{n+1} + d_1 y_n + d_0 y_{n-1} + h^2[\hat{d}_4 f_{n+4} + \hat{d}_3 f_{n+3} + \hat{d}_2 f_{n+2} + \hat{d}_1 f_{n+1} + \hat{d}_0 f_n], \quad (25)$$

where the coefficients for the methods (22)-(25) are respectively obtained as follows;

$$a_0 = \frac{29423}{147456}, a_1 = \frac{7441}{294912}, a_2 = \frac{228625}{294912}, \hat{a}_0 = \frac{3826211}{70778880},$$
$$\hat{a}_1 = -\frac{349231}{4423680}, \hat{a}_2 = \frac{656087}{35389440}, \hat{a}_3 = \frac{1933}{368640}, \hat{a}_4 = \frac{51509}{70778880} \quad (26)$$

and principal local truncation error given by $\frac{1278967}{71345111040} y^{(8)}(x)h^8 + O(h^9)$.

$$b_0 = \frac{175201}{147456}, b_1 = \frac{82847}{294912}, b_2 = \frac{138337}{294912}, \hat{b}_0 = \frac{22121411}{70778880},$$
$$\hat{b}_1 = \frac{322771}{4423680}, \hat{b}_2 = \frac{444407}{35389440}, \hat{b}_3 = \frac{247}{92160}, \hat{b}_4 = \frac{21269}{70778880} \quad (27)$$

$$c_0 = \frac{5788393}{18874368}, c_1 = \frac{1289495}{37748736}, c_2 = \frac{24882455}{37748736}, \hat{c}_0 = \frac{135287945}{1811939328},$$
$$\hat{c}_1 = -\frac{10740691}{113246208}, \hat{c}_2 = \frac{21808709}{905969664}, \hat{c}_3 = \frac{129623}{18874368}, \hat{c}_4 = \frac{1732991}{1811939328} \quad (28)$$

$$d_0 = \frac{26460343}{18874368}, d_1 = \frac{4210505}{37748736}, d_2 = \frac{19382455}{37748736}, \hat{d}_0 = \frac{58167674}{51811939328},$$
$$\hat{d}_1 = \frac{10285691}{113246208}, \hat{d}_2 = \frac{18168709}{905969664}, \hat{d}_3 = \frac{97123}{18874368}, \hat{d}_4 = -\frac{121299}{11811939328} \quad (29)$$

and principal local truncation error as $\frac{969182695}{3652869685248} y^{(8)}(x)h^8 + 0(h^9)$.

The stability region of the new method is described by Figure 1, which shows an indefinite progression along the positive real axis satisfying Definition 1.5.

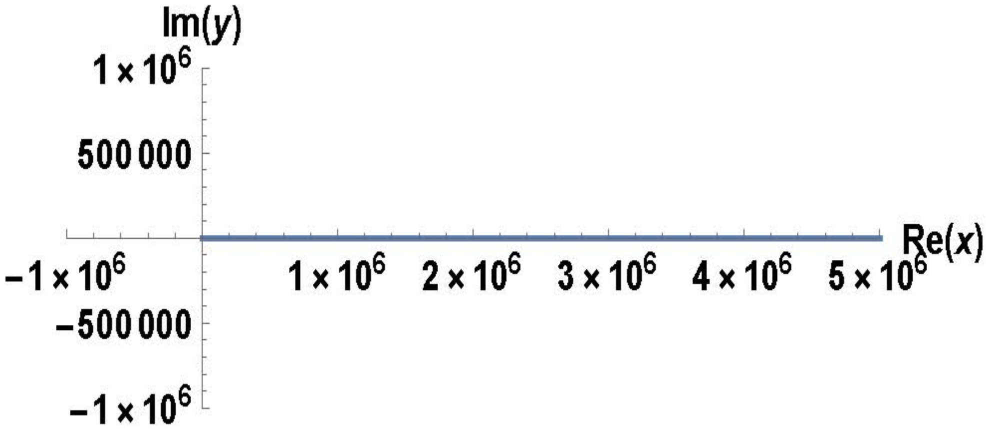


Figure 1 The stability plot of the new scheme

3. IMPLEMENTATION OF THE PROPOSED METHODS

In this section, the main task shall be the resolution of the implicitness that is naturally imposed on (2) by P-stability and carry out numerical experiment to show the accuracy and efficiency of the new algorithms. The IVP (1) is assumed to satisfy the Lipschitz condition with reference to $y(x)$ for all $x \in [a,b]$, i.e., $\| f(x,y) - f(x,y*) \| \leq L \| y - y* \|$, where L is the Lipschitz constant.

The Newton-Raphson's iterative method in [1,4] has been adopted in our implementation of the new method to solve the following second order IVP.

Example 1 (An Orbital Problem: Source [10])

We consider a periodic problem defined by

$$y'' = -y + 0.001e^{ix} \quad (30)$$

$$y(0) = 1, \quad y'(0) = 0.9995i, \quad i^2 = -1,$$

which has analytical solution as

$$y(x) = u(x) + v(x)$$

$$\begin{aligned} u(x) &= \cos x + 0.0005x\sin x \\ v(x) &= \sin x - 0.0005x\cos x \end{aligned} \quad (31)$$

The initial value problem (30) describes a motion on a circular orbit perturbed in the complex plane such that the point $y(x)$ slowly spirals outward and its distance from the origin at any given time x is described by

$$\Psi(x) = \sqrt{u^2(x) + v^2(x)} \quad (32)$$

Using the predictor

$$y_n - 2y_{n+1} + y_n = h^2 f_{n+1}, \quad (33)$$

the system of equation in (30) can be approximated in the interval [0, 40] which corresponds to 20 orbits of the points $y(x)$. The integration is then carried out with uniform mesh sizes

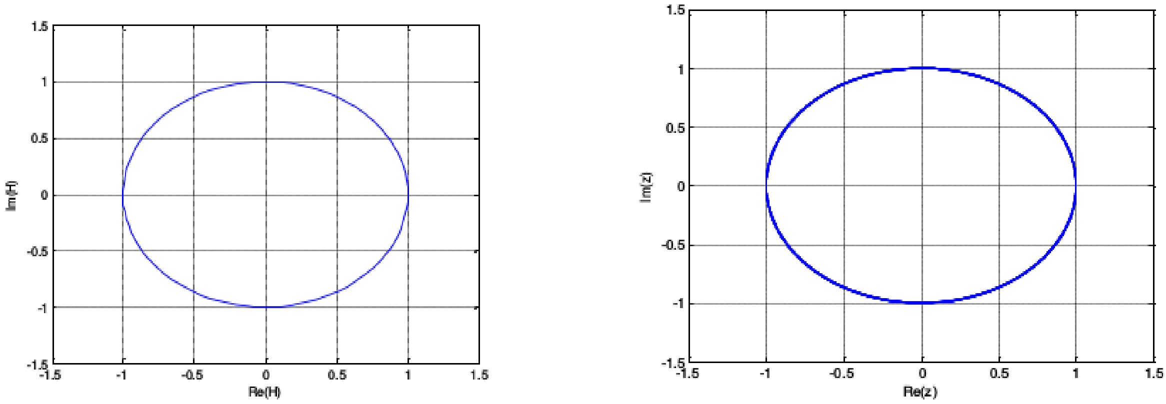
$$h = \frac{\pi}{2^q}, q = 3(1)10 \quad (34)$$

The numerical performance of our new algorithm are presented in Table 1.

Table 1: Numerical Experiment

q	h	New Method (20)	Error	Method in [7]	Error	Method in [4]	Error
3	$\frac{\pi}{2^3}$	1.001987	3.63E-5	1.002034	6.20E-5	–	–
4	$\frac{\pi}{2^4}$	1.001979	8.23E-6	1.002003	3.09E-5	1.004118	2.15E-3
5	$\frac{\pi}{2^5}$	1.001976	9.38E-6	1.001987	3.63E-5	1.002856	8.84E-4
6	$\frac{\pi}{2^6}$	1.001974	6.78E-6	1.001979	7.70E-6	1.002400	4.3E-4
7	$\frac{\pi}{2^7}$	1.001973	7.78E-6	1.001975	3.84E-6	–	–
8	$\frac{\pi}{2^8}$	1.001972	8.98E-7	1.001973	1.92E-6	–	–
9	$\frac{\pi}{2^9}$	1.001972	8.98E-7	1.001972	9.62E-7	1.002057	8.50E-5
10	$\frac{\pi}{2^{10}}$	1.001972	8.98E-7	1.001972	4.89E-7	–	–

The new algorithm (20) is applied to investigate the behaviour of problem (30). Numerical simulation comparing with its analytical solution is shown in Figure 2:



Analytical Solution Numerical Solution

Figure 2 Numerical Simulation of an orbital problem comparing with its analytical solution

Example 2 (Stiff Oscillatory Problem: Source [11])

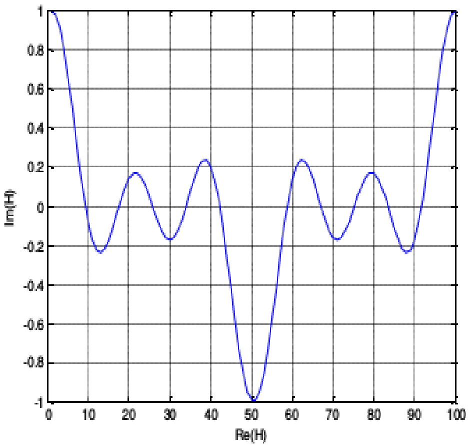
Consider the stiff oscillatory problem defined by

$$y''(t)+m^2y(t)=8cos(t)+\frac{2}{3}cos(3t), \quad y(0)=1, y'(0)=0 \text{, with } m=5 \qquad (35)$$

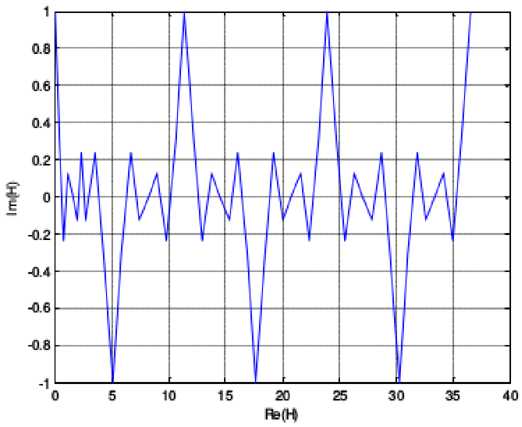
whose exact solution is

$$y(t)=\frac{1}{3}(cost+cos3t+cos5t). \qquad (36)$$

In a similar manner, we implement the system (35) using meshsize $h=\frac{\pi}{8}$ at $t=10\pi$



Analytical Solution



Numerical Solution

Figure 3 Numerical Simulation of Stiff Oscillatory Problem comparing with its analytical solution

4. CONCLUSIONS

In this paper, a generalized hybrid parameter is defined from which we obtained hybrid parameters to construct higher order P-stable Super-implicit method for the numerical solution of special second order IVP with oscillating solutions. The new algorithms derived using this generalization possess higher order of accuracy and stronger stability properties that are illustrated in Table 1 and Figures 1, 2 and 3.

ACKNOWLEDGEMENT

The authors would like to appreciate Covenant University Centre for Research, Innovation and Discovery (CUCRID) for sponsoring this publication.

REFERENCES

[1] Akinlabi, G.O., Adeniyi, R.B., Owoloko, E.A., (2018). The solution of boundary value problems with mixed boundary conditions via boundary value methods. Int. J. Circ., Syst. Signal Process., 2: 1-6.

[2] Anake, T.A., Bishop, S.A., Agboola, O. O., (2015). On a Hybrid Numerical Algorithm for the Solutions of Higher Order Ordinary Differential Equations. TWMS J. Pure Appl. Math., 6(2): 186-193.

[3] Anake, T.A., Awoyemi, D.O. and Adesanya, A.O., (2012). One-Step Implicit Hybrid Block Method for the Direct Solution of General Second Order Ordinary Differential Equations. IAENG Int. J. of Appl. Math., 42(4): 224-228.4.

[4] Cash J.R., (1981). High order P-stable Formulae for the Numerical Integration of Periodic Initial Value Problems. J. Numer. Math., 37: 355-370.

[5] Felix I. C., and Okuonghae R. I., (2018). On the Construction of P-stable Hybrid Multi-step Methods for second order ODEs. Far East J. Appl. Math., 99(3): 259-273.6. 6.

[6] Fukushima T., (1998). “Symmetric multistep methods, revisited”, In Prec. of the 30 tu Syrup. on Celestial Mechanics, 4-6 March 1998, Hayama, Kanagawa, Japan (Edited by T. Fukushima, T. Ito, T. Fuse and H. Umehara): 229-247.7.

[7] Ibrahim O.M. and Ikhile M.N.O. (2017). Highly stable Super-implicit Hybrid methods for second order IVP, J.of Science. Tech., and Pub. Pol., 4: 100-106.

[8] Ita, B.I., Ehi-Eromosele, C.O., Edobor-Osoh, A., Ikeuba, A.I., Anake, T.A., (2014) Solutions of the Schro¨dinger Equation with Inversely Quadratic Effective Plus Mie-Type Potential using Nikiforov-Uvarov Method. AIP Conference Proceedings 1629,235); doi: 10.1063/1.4902278.

[9] Jain M.K., Jain R.K. and Anantha Krishnaiah U., (1979). P-stable Methods for Periodic Initial Value Problems of Second Order Differential Equations. BIT, 19:347-355.

[10] Lambert J. D. and Watson I., (1976). Symmetric Multistep Methods For Periodic Initial Value Problems, J. Inst. Math. Appls., 18: 189-202.

[11] Mehdizadeh Khalsaraei1 M. and Molayi M., (2015). P-stable Hybrid Super-Implicit Methods for Periodic Initial Value Problems, J. Math. Comput. Sci., 15: 129–136.

[12] Neta B., (2005), P-stable Symmetric Super-Implicit Methods for Periodic Initial Value Problems, Comput. Math. Appl., 50: 701–705.

[13] Oghonyon, J.G., Omoregbe, N.A., Bishop, S.A., (2016). Implementing an order six implicit block multistep method for third order ODEs using variable step size approach, Global J. Pure Appl. Math., 2(2): 1653-1646.