ON DISTORTION ENERGY THEORY IN HIGH CYCLE MULTI-AXIAL FATIGUE

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ABSTRACT

Structural members when subjected to cyclic normal stresses in different directions or combined action of cyclic normal and shear stresses fail due to multi-axial fatigue. Distortion energy theory is recommended for investigation of multi-axial fatigue in ductile materials whereas normal stress theory is opted for brittle materials. The paper presents a review on application of distortion energy theory that is based upon the static state of stress and the modified Goodman relationships. Some problems are solved to demonstrate the procedure. The results are validated with the help of equivalent stress method and Goodman’s basic criterion.

Keywords: Cyclic stress, Distortion energy theory, Goodman’s criterion, High cycle fatigue, Multi-axial fatigue, Principal stress

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$p$ Pressure
$P$ Axial load
$t$ Thickness
$T$ Twisting moment
$U_d$ Distortion energy per unit volume
$\nu$ Poisson's ratio
$\sigma$ Normal stress/Uni - axial stress
$\tau$ Shear stress

$\sigma_{1;2;3}$ Principal stresses
$\sigma_f$ Fatigue strength
$\sigma_f$ Endurance or fatigue limit
$\sigma_y$ Yield strength
$\sigma_y'$ Cyclic yield strength
$\sigma_{tu}$ Ultimate tensile strength

Subscripts
$\alpha$ Amplitude
$eq$ Equivalent value
$m$ Mean value
$max$ Maximum value
$min$ Minimum value

1. INTRODUCTION

Many real life components like rotating shafts, connecting links, turbine blades, aircraft members, automotive parts etc. operate under multi-axial cyclic stress in high cycle conditions. Cases of normal stresses in different directions, pure shear and combination of normal and shear stresses all fall within the purview of multi-axial fatigue. The worst possible case could be the normal stresses acting in all three cartesian directions coupled with shear stresses in the associated planes. Analysis of multi-axial fatigue calls for determination of cyclic principal stresses with maximum, minimum and mean magnitudes that are then used in different theories viz. maximum normal stress multi-axial fatigue failure theory, maximum shear stress multi-axial fatigue failure theory and distortion energy multi-axial fatigue failure theory. These theories involve use of modified Goodman’s relationships to account for the effect of non-zero mean stress. Distortion energy theory is recommended for ductile materials whereas maximum normal stress theory is suitable for brittle materials.

Papadopoulos et al. [1] assessed some commonly used high-cycle fatigue criteria to check their predictive capabilities against synchronous sinusoidal out-of-phase bending and torsion experimental results. Lui and Mahadevan [2] proposed a new high-cycle fatigue criterion based on the critical plane approach. Unlike most of other multiaxial fatigue criteria based on the critical plane approach, the critical plane was found to be directly correlated with the fatigue fracture plane. A strain energy based method was developed by Emuakpor et al. [3] to predict the fatigue life of a structure subjected to either shear or bi-axial bending loads at various stress ratios. The developed framework consisted of two elements – life prediction method that calculated shear fatigue life cycles and multi-axial life prediction method capable of calculating bi-axial fatigue life cycles. Ellyin and Golos [4] suggested a multiaxial fatigue failure criterion based on the strain energy density damage law. The proposed criterion was hydrostatic pressure sensitive; included the effect of the mean stress, and was applied to materials which did not obey the idealized Masing type description. The material constants were evaluated from two simple test results, e.g., uniaxial tension and torsion fatigue tests. The predicted results were compared with biaxial tests and the agreement was found to be fairly good. A desirable feature of this criterion was its unifying nature for both short and long cyclic lives. It was also consistent with the crack initiation and propagation phases of the...
fatigue life, in the sense that both of these phases could be related to the strain energy density either locally or globally. Atzori et al. [5] determined multi-axial fatigue strength of notched specimens made of C40 carbon steel (normalized state), subjected to combined tension and torsion loading, both in-phase and out-of-phase. Zengliang et al. [6] tested V-notched specimens under two nominal load ratios, \( R = -1 \) and 0, while keeping the bi-axiality ratio constant and equal to the unity. Based on the tension–compression, torsion, and axial-torsion fatigue experiments conducted on notched shaft specimens made of 16MnR steel, local stress fatigue life prediction approaches were evaluated. The cyclic elastic–plastic deformation of the material at the notch was analyzed using the finite element (FE) and some approximate methods. Two critical plane multiaxial fatigue criteria were used to predict the fatigue lives. The predicted fatigue lives based on the FE stress analysis agreed well with the experimental observations and predictions made using the local stress–strain results obtained from the approximate methods did not agree well with the experimental results. Fatemi and Socie [7] proposed modification to Brown and Miller's critical plane approach to predict multiaxial fatigue life under both in-phase and out-of-phase loading conditions. The components of this modified parameter consisted of the maximum shear strain amplitude and the maximum normal stress on the maximum shear strain amplitude plane. Additional cyclic hardening developed during out-of-phase loading was included in the normal stress term. Also, the mathematical formulation of this new parameter was such that variable amplitude loading could be accommodated. Experimental results from tubular specimens made of 1045 HR steel under in-phase and 90° out-of-phase axial-torsional straining using both sinusoidal and trapezoidal wave forms were correlated within a factor of about two. Available Inconel 718 axial-torsional data including mean strain histories were satisfactorily correlated using the aforementioned parameter. Andrea C. and Andrea S. [8] reviewed some multiaxial high-cycle fatigue criteria based on the so-called plane approach. The critical plane orientation was correlated with the averaged principal stress directions deduced through the weight function method and new fatigue failure criterion was proposed. The results derived by applying the present criterion and the other critical plane criteria analyzed were compared with experimental data related to different brittle (hard) metals under in-phase or out-of-phase sinusoidal biaxial normal and shear stress states. A method for estimating high-cycle fatigue strength under multiaxial loading conditions was reviewed by Susmel and Lazzarin [9] where in the physical interpretation of the fatigue damage was based on the theory of cyclic deformation in single crystals. Fatigue life estimates were carried out by means of a modified Wöhler curve which could be applied to both smooth and blunt notched components, subjected to either in-phase or out-of-phase loads. The modified Wöhler curve plotted the fatigue strength in terms of the maximum macroscopic shear stress amplitudes, the reference plane–where such amplitudes had to be evaluated–being thought of as coincident with the fatigue micro crack initiation plane. The position of the fatigue strength curve depended on the stress component normal to such a plane and the phase angle as well. A new failure criterion by Mcdiarmid [10] was based on a critical plane approach where fatigue strength was considered as a function of shear stress amplitude and the maximum normal stress on the critical plane of maximum shear stress amplitude.

This paper presents a review on role and application of distortion energy theory in multi-axial fatigue analysis of ductile member subjected to constant amplitude load in high cycle conditions. The theory represents the combination of static state of stress and the modified Goodman relationships. Some interesting problems are solved to demonstrate the procedure. The results are validated with the help of equivalent stress method and Goodman’s basic criterion.
2. DISTORTION ENERGY MULTI-AXIAL FATIGUE FAILURE THEORY

For a general 3D stress state defined by $\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$, the three principal stresses $\sigma_1, \sigma_2$ and $\sigma_3$ are the roots of the cubic equation $\sigma^3 - I_1 \sigma^2 + I_2 \sigma - I_3 = 0$ where the stress invariants, $I_1, I_2$ and $I_3$ are written as $I_1 = \sigma_x + \sigma_y + \sigma_z$, $I_2 = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$ and $I_3 = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$. For 2D stress state represented as $\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix}$, the solution assumes the well-known form, $\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{xy}^2}$; $\sigma_3 = 0$. The expression for distortion energy per unit volume of 3D stress system from elementary principles is given by Eq. (1)

$$U_d = \frac{1 + \nu}{6E} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]$$

Using this basic expression, the following equations are obtained for maximum, mean and minimum values of distortion energy.

$$U_{d,max} = \frac{1 + \nu}{6E} \left[ (\sigma_{1,max} - \sigma_{2,max})^2 + (\sigma_{2,max} - \sigma_{3,max})^2 + (\sigma_{3,max} - \sigma_{1,max})^2 \right]$$

$$U_{d,m} = \frac{1 + \nu}{6E} \left[ (\sigma_{1,m} - \sigma_{2,m})^2 + (\sigma_{2,m} - \sigma_{3,m})^2 + (\sigma_{3,m} - \sigma_{1,m})^2 \right]$$

$$U_{d,min} = \frac{1 + \nu}{6E} \left[ (\sigma_{1,min} - \sigma_{2,min})^2 + (\sigma_{2,min} - \sigma_{3,min})^2 + (\sigma_{3,min} - \sigma_{1,min})^2 \right]$$

Eq. (2) to (4) take the following forms for equivalent uni-axial state of cyclic stress

$$U_{d,max} = \frac{1 + \nu}{3E} \left[ (\sigma_{max})^2 \right] \text{ or } \sigma_{max} = \frac{3E}{1 + \nu} U_{d,max}$$

$$U_{d,m} = \frac{1 + \nu}{3E} \left[ (\sigma_{m})^2 \right] \text{ or } \sigma_{m} = \frac{3E}{1 + \nu} U_{d,m}$$

$$U_{d,min} = \frac{1 + \nu}{3E} \left[ (\sigma_{min})^2 \right] \text{ or } \sigma_{min} = \frac{3E}{1 + \nu} U_{d,min}$$

Eq. (5) to (7) are used in modified Goodman’s relationships [11] for design analysis and prediction of failure. Each failure condition is followed by the region of validity depending upon the mean stress, $\sigma_m$. The relations are:

i) $\sigma_{max} - 2\sigma_m \geq \sigma_{ys}$ for $-\sigma_{ys} \leq \sigma_m \leq \left(\sigma_f - \sigma_{ys}\right)$

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ii) \( \sigma_{\text{max}} - \sigma_m \geq \sigma_f \) for \( (\sigma_f - \sigma_{\text{ys}}) \leq \sigma_m \leq 0 \)  

iii) \( \sigma_{\text{max}} - (1-r)\sigma_m \geq \sigma_f \) for \( 0 \leq \sigma_m \leq \left( \frac{\sigma_{\text{ys}} - \sigma_f}{1-r} \right) \)  

iv) \( \sigma_{\text{max}} \geq \sigma_{\text{ys}} \) for \( \left( \frac{\sigma_{\text{ys}} - \sigma_f}{1-r} \right) \leq \sigma_m \leq \sigma_{\text{ys}} \)  

where \( r = \frac{\sigma_f}{\sigma_{\text{uts}}} \). Fatigue strength, \( \sigma_f \), in the equations is replaced by fatigue or endurance limit, \( \sigma_e \), in case of design for infinite life. Eq 8) and 9) correspond to \(-ve\) \( \sigma_m \) while Eq 10) and 11) are meant for \(+ve\) \( \sigma_m \). Since distortion energy is always a positive quantity, it cannot be used in specifying the region of validity of mean stress in Eq. (8) to (11). Instead all the equations are evaluated separately and the design is based on the most critical equation.


Problems concerning members subjected to various types of multi-axial cyclic loads are defined. In each problem, the maximum, mean and minimum values of principal stress are first found in all cartesian directions from known applied cyclic stresses. The magnitudes of principal stresses are then used to obtain maximum and mean values of distortion energy that in turn is followed by determination of maximum and mean values of equivalent uni-axial stress. The desired parameters are finally obtained by using uni-axial stress values in modified Goodman’s relationships. The solutions of the problems are as follows:-

3.1: Refer Fig.1. An aluminum alloy bar of solid circular cross section is to be subjected to a cyclic axial load that ranges from 5000 lb tension to 10000 lb tension. The material has an ultimate tensile strength of 100000 psi, yield strength of 80000 psi, mean fatigue strength of 40000 psi at \( 10^5 \) cycles and elongation of 8% in 2 inches. Calculate the bar diameter, \( d \), that should be used to produce failure in \( 10^5 \) cycles. Assume factor of safety as 2.

Solution:-

Given: \( P_{\text{min}} = 5000 \) lb, \( P_{\text{max}} = 10000 \) lb, \( \sigma_{\text{uts}} = 100000 \) psi, \( \sigma_{\text{ys}} = 80000 \) psi, \( \sigma_f = 40000 \) psi at \( 10^5 \) cycles

\( FOS = 2 \)

Since aluminum alloy with high % elongation is a ductile material, distortion energy theory is used.

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Normal stress, \( \sigma_{x,\text{max}} = \frac{10000}{\pi d^2} = \frac{12726.7}{d^2} \), \( \sigma_{x,\text{min}} = \frac{5000}{\pi d^2} = \frac{6363.3}{d^2} \)

Principal stresses are \( \sigma_{1,\text{max}} = \sigma_{x,\text{max}} = \frac{12726.7}{d^2} \), \( \sigma_{1,\text{min}} = \sigma_{x,\text{min}} = \frac{6363.3}{d^2} \), \( \sigma_{1,\text{m}} = \frac{\sigma_{1,\text{max}} + \sigma_{1,\text{min}}}{2} = \frac{9545}{d^2} \)

\( \sigma_{2,\text{max}} = \sigma_{2,\text{min}} = \sigma_{2,m} = 0 \); \( \sigma_{3,\text{max}} = \sigma_{3,\text{min}} = \sigma_{3,m} = 0 \)

\[
U_{d,\text{max}} = \left[ \frac{1 + \nu}{6E} \right] \left[ (\sigma_{1,\text{max}} - \sigma_{2,\text{max}})^2 + (\sigma_{2,\text{max}} - \sigma_{3,\text{max}})^2 + (\sigma_{3,\text{max}} - \sigma_{1,\text{max}})^2 \right] = \left[ \frac{1 + \nu}{6E} \right] \times 2 \times \left( \frac{12726.7}{d^2} \right)^2
\]

Equivalent uni-axial maximum stress, \( \sigma_{\text{max}} = \sqrt{\frac{3E}{1+\nu} \frac{U_{d,\text{max}}}{d^2}} = \frac{12726.7}{d^2} \)

\[
U_{d,m} = \left[ \frac{1 + \nu}{6E} \right] \left[ (\sigma_{1,m} - \sigma_{2,m})^2 + (\sigma_{2,m} - \sigma_{3,m})^2 + (\sigma_{3,m} - \sigma_{1,m})^2 \right] = \left[ \frac{1 + \nu}{6E} \right] \times 2 \times \left( \frac{9545}{d^2} \right)^2
\]

Equivalent uni-axial mean stress, \( \sigma_{m} = \frac{\frac{3E}{1+\nu} U_{d,m}}{d^2} = \frac{9545}{d^2} \)

\[
r = \frac{\sigma_f}{\sigma_{\text{uts}}} = \frac{40000}{10000} = 0.4
\]

Modified Goodman’ s failure equations are

i) \( \sigma_{\text{max}} - 2\sigma_m \geq \frac{\sigma_{\text{ys}}}{\text{FOS}} \)

or \( \frac{12726.7}{d^2} - 2 \times \frac{9545}{d^2} = \frac{80000}{2} \); The condition is not valid since LHS is always - ve for + ve value of \( d \)

ii) \( \sigma_{\text{max}} - \sigma_m \geq \frac{\sigma_f}{\text{FOS}} \)

or \( \frac{12726.7}{d^2} - \frac{9545}{d^2} = \frac{40000}{2} \)

or \( d = 0.398 \) inches

iii) \( \sigma_{\text{max}} - (1-r)\sigma_m \geq \frac{\sigma_f}{\text{FOS}} \)

or \( \frac{12726.7}{d^2} - (1-0.4) \frac{9545}{d^2} = \frac{40000}{2} \)

or \( d = 0.59 \) inches

iv) \( \sigma_{\text{max}} \geq \frac{\sigma_{\text{ys}}}{\text{FOS}} \)

or \( \frac{12726.7}{d^2} = \frac{80000}{2} \)

or \( d = 0.56 \) inches

Select highest value, \( d = 0.59 \) in (Ans)

Equivalent stress method: Since the problem deals with uni-axial stress state, the equivalent stress is same as the applied stress. Equivalent stress [12] is written as
$$\sigma_{eq}^2 = \frac{1}{2} \left[ (\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6\tau_{xy}^2 + 6\tau_{yz}^2 + 6\tau_{zx}^2 \right]$$

In the present case, $\sigma_{eq} = \sigma_x$. Therefore $\sigma_{eq,a} = \sigma_{x,a}$ and $\sigma_{eq,m} = \sigma_{x,m}$

where $\sigma_{x,a} = \frac{\sigma_{x,\text{max}} - \sigma_{x,\text{min}}}{2} = \frac{318.17\text{ psi}}{d^2}; \sigma_{x,m} = \frac{\sigma_{x,\text{max}} + \sigma_{x,\text{min}}}{2} = \frac{9545\text{ psi}}{d^2}$

Use of Goodman’s basic criterion, $\frac{\sigma_{eq,a} \times \text{FOS}}{\sigma_f} + \frac{\sigma_{eq,m} \times \text{FOS}}{\sigma_{uts}} = 1$, also results in $d = 0.59$ in. Fulfillment of the conditions $\sigma_{x,\text{max}} < \sigma_{ys}$ and $\sigma_{eq,a} < \sigma'_{ys}$ confirms high cycle fatigue. Hence use of Goodman’s relations is valid.

3.2: Refer Fig. 2. A thin cylindrical pressure vessel, closed at both ends, has an inside diameter of 3 inches to meet design requirements. The material used is steel alloy with following properties:- Endurance limit = 100000 psi, Ultimate tensile strength = 250000 psi, yield strength 200000 psi. The vessel is pressurized cyclically from 0 to 15000 psi once a minute continuously for 10 years. Using factor of safety of 1.5, determine the outside diameter.

Solution:-

Given-

$d = 3\text{ in}, p_{\text{min}} = 0, p_{\text{max}} = 15000\text{ psi}, \sigma_{uts} = 250000\text{ psi}, \sigma_{ys} = 200000\text{ psi}, \sigma_f = 100000\text{ psi}, \text{FOS} = 1.5$

Since alloy steel is a ductile material, distortion energy theory is used. Number of cycles the material has to withstand $60 \times 24 \times 365 \times 10 = 5256000$ cycles that is greater than $10^6$ cycles. Therefore the vessel needs to have infinite life that necessitates use of endurance limit in design process. Hoop (Circumferential) and longitudinal stresses act as principal stresses in absence of shear stress.

![Figure 2 Member of Case 3.2](http://www.iaeme.com/IJMET/index.asp)
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Hoop stress, \( \sigma_{x,\text{max}} = \frac{pd}{2t} = \frac{15000 \times 3}{2t} = \frac{22500}{t} \), Longitudinal stress due to closedends,
\[ \sigma_{y,\text{max}} = \frac{pd}{4t} = \frac{15000 \times 3}{4t} = \frac{11250}{t} \]
Principal stresses are \( \sigma_{1,\text{max}} = \sigma_{x,\text{max}} = \frac{22500}{t}, \sigma_{1,\text{min}} = 0, \sigma_{2,\text{max}} = \frac{\sigma_{2,\text{max}} + \sigma_{2,\text{min}}}{2} = \frac{5625}{t} \)
\[ \sigma_{3,\text{max}} = \sigma_{3,\text{min}} = 0 \]

\[ U_{d,\text{max}} = \left[ \frac{1 + \nu}{6E} \right] \left[ (\sigma_{1,\text{max}} - \sigma_{2,\text{max}})^2 + (\sigma_{2,\text{max}} - \sigma_{3,\text{max}})^2 + (\sigma_{3,\text{max}} - \sigma_{1,\text{max}})^2 \right] = \left[ \frac{1 + \nu}{6E} \right] \times 6 \times \left( \frac{11250}{t} \right)^2 \]

Equivalent uni - axial maximum stress, \( \sigma_{\text{max}} = \sqrt{\frac{3E}{1 + \nu}} U_{d,\text{max}} = \sqrt{3} \times \frac{11250}{t} \)

\[ U_{d,\text{m}} = \left[ \frac{1 + \nu}{6E} \right] \left[ (\sigma_{1,\text{m}} - \sigma_{2,\text{m}})^2 + (\sigma_{2,\text{m}} - \sigma_{3,\text{m}})^2 + (\sigma_{3,\text{m}} - \sigma_{1,\text{m}})^2 \right] = \left[ \frac{1 + \nu}{6E} \right] \times 6 \times \left( \frac{5625}{t} \right)^2 \]

Equivalent uni - axial mean stress, \( \sigma_{\text{m}} = \sqrt{\frac{3E}{1 + \nu}} U_{d,\text{m}} = \sqrt{3} \times \frac{5625}{t} \)

\[ r = \frac{\sigma_r}{\sigma_{\text{uts}}} = \frac{100000}{250000} = 0.4 \]

Modified Goodman’ s failure equations are

i) \( \sigma_{\text{max}} - 2\sigma_m \geq \frac{\sigma_y}{FOS} \)

or \( \sqrt{3} \times \frac{11250}{t} - 2 \times \sqrt{3} \times \frac{5625}{t} = \frac{200000}{1.5} \); The condition is not valid as LHS is zero.

ii) \( \sigma_{\text{max}} - \sigma_m \geq \frac{\sigma_r}{FOS} \)

or \( \sqrt{3} \times \frac{11250}{t} - \sqrt{3} \times \frac{5625}{t} = \frac{100000}{1.5} \)

or \( t = 0.146 \) in

iii) \( \sigma_{\text{max}} - (1 - r)\sigma_m \geq \frac{\sigma_r}{FOS} \)

or \( \sqrt{3} \times \frac{11250}{t} - (1 - 0.4) \times \sqrt{3} \times \frac{5625}{t} = \frac{100000}{1.5} \)

or \( t = 0.2 \) in

iv) \( \sigma_{\text{max}} \geq \frac{\sigma_y}{FOS} \)

or \( \sqrt{3} \times \frac{11250}{t} = \frac{200000}{1.5} \)

or \( t = 0.146 \) in

Select highest value, \( t = 0.2 \) in. Outer diameter of the vessel is \( 3 + 2t = 3.4 \) in (Ans)
Equivalent stress method

Equivalent stress in present case, \( \sigma_{eq} = \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y} \)

\[
\sigma_{eq,a} = \sqrt{\sigma_{x,a}^2 + \sigma_{y,a}^2 - \sigma_{x,a} \sigma_{y,a} = \frac{\left(11250}{t}\right)^2 + \frac{\left(5625}{t}\right)^2 - \left(11250 \times \frac{\left(5625}{t}\right)^2}{t} = \sqrt{3} \times \frac{5625}{t}
\]

\[
\sigma_{eq,m} = \sqrt{\sigma_{x,m}^2 + \sigma_{y,m}^2 - \sigma_{x,m} \sigma_{y,m} = \frac{\left(11250}{t}\right)^2 + \frac{\left(5625}{t}\right)^2 - \left(11250 \times \frac{\left(5625}{t}\right)^2}{t} = \sqrt{3} \times \frac{5625}{t}
\]

where \( \sigma_{x,a} = \frac{\sigma_{x,max} - 0}{2} \), \( \sigma_{y,a} = \frac{\sigma_{y,max} - 0}{2} \), \( \sigma_{x,m} = \frac{\sigma_{x,max} + 0}{2} \) and \( \sigma_{y,m} = \frac{\sigma_{y,max} + 0}{2} \)

Use of Goodman’s basic criterion, \( \frac{\sigma_{eq,a} \times FOS}{\sigma_e} + \frac{\sigma_{eq,m} \times FOS}{\sigma_{uts}} = 1 \), results in \( t = 0.204 \) in that is close to 0.2 in. Fulfillment of the conditions \( \sigma_{max} < \sigma_{ys} \) and \( \sigma_{eq,a} < \sigma_{ys} \) confirms high cycle fatigue.

3.3: Refer Fig. 3. A power transmission shaft of circular cross section is made of hot rolled 1020 steel with ultimate tensile strength of 65000 psi, yield strength of 43000 psi and endurance limit of 32000 psi. The shaft is to transmit 85 HP steadily at rotational speed of 1800 rpm with no fluctuations in torque or speed. At the critical location mid span between bearings, the shaft is also subjected to bending moment of 2000 in-lb. Determine the required shaft diameter to provide infinite life. Assume factor of safety as unity

![Figure 3 Member of Case 3.3](image)

Solution

Given : HP = 85, \( N = 1800 \) rpm, \( M = 2000 \) in-lb, \( \sigma_{uts} = 65000 \) psi, \( \sigma_{ys} = 43000 \) psi, \( \sigma_e = 32000 \) psi, \( FOS = 1 \)

Since it is the case of rotating shaft, the bending load shall be completely reversible in nature. Also as steel is a ductile material, distortion energy theory is used.
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\[ HP = \frac{2\pi NT}{60 \times 746} \] where \( N \) is in rpm and twisting moment, \( T \), is in N \( \cdot \) m

\[ T = 336.24 \text{ N} \cdot \text{m} = 2971.74 \text{ in} \cdot \text{lb} \], shear stress at surface point, \( \tau_{\text{sz},\text{max}} = \frac{16T}{\pi d^3} = 15128 \cdot 1 \)

\[ M = 2000 \text{ in} \cdot \text{lb} \], bending stress at surface point, \( \sigma_{x,\text{max}} = \frac{32M}{\pi d^3} = 20362 \cdot 7 \), \( \sigma_{x,\text{min}} = \frac{32M}{\pi d^3} = -20362 \cdot 7 \)

Principal stresses for combination of bending and torsion are given by

\[ \frac{16}{\pi d^3} \left[ M \pm \sqrt{M^2 + T^2} \right] \]

For positive peak value of \( M \)

\[ \sigma_{1,\text{max}} = \frac{16}{\pi d^3} \left[ M + \sqrt{M^2 + T^2} \right] = \frac{28413 \cdot 4}{d^3}, \sigma_{2,\text{max}} = \frac{16}{\pi d^3} \left[ M - \sqrt{M^2 + T^2} \right] = -8050 \]

For negative peak value of \( M \)

\[ \sigma_{1,\text{min}} = -\frac{16}{\pi d^3} \left[ M + \sqrt{M^2 + T^2} \right] = -\frac{28413 \cdot 4}{d^3}, \sigma_{2,\text{min}} = -\frac{16}{\pi d^3} \left[ M - \sqrt{M^2 + T^2} \right] = 8050 \]

\[ \sigma_{3,\text{max}} = \sigma_{3,\text{min}} = \sigma_{3,m} = 0 \]

\[ U_{d,m} = \left[ \frac{1 + \nu}{6E} \right] \left[ (\sigma_{1,\text{max}} - \sigma_{2,\text{max}})^2 + (\sigma_{2,\text{max}} - \sigma_{3,\text{max}})^2 + (\sigma_{3,\text{max}} - \sigma_{1,\text{max}})^2 \right] \]

Equivalent uni - axial maximum stress, \( \sigma_{\text{max}} = \sqrt{\frac{3E}{1 + \nu}} U_{d,max} = \frac{33179 \cdot 5}{d^3} \)

\[ U_{d,m} = \left[ \frac{1 + \nu}{6E} \right] \left[ (\sigma_{1,m} - \sigma_{2,m})^2 + (\sigma_{2,m} - \sigma_{3,m})^2 + (\sigma_{3,m} - \sigma_{1,m})^2 \right] = 0 \]

Equivalent uni - axial mean stress, \( \sigma_{m} = \sqrt{\frac{3E}{1 + \nu}} U_{d,m} = 0 \)

\[ r = \frac{\sigma_{e}}{\sigma_{uts}} = \frac{32000}{65000} = 0.49 \]

Modified Goodman’s failure equations are

i) \[ \sigma_{\text{max}} - 2\sigma_{m} \geq \frac{\sigma_{s}}{FOS} \]

or \[ \frac{33179 \cdot 5}{d^3} = \frac{43000}{1} \Rightarrow d = 0.91 \text{ in} \]

ii) \[ \sigma_{\text{max}} - \sigma_{m} \geq \frac{\sigma_{e}}{FOS} \]

or \[ \frac{33179 \cdot 5}{d^3} = \frac{32000}{1} \Rightarrow d = 1.01 \text{ in} \]

iii) \[ \sigma_{\text{max}} - (1 - r)\sigma_{m} \geq \frac{\sigma_{e}}{FOS} \]

or \[ \frac{33179 \cdot 5}{d^3} = \frac{32000}{1} \Rightarrow d = 1.01 \text{ in} \]
iv) \( \sigma_{\text{max}} \geq \frac{\sigma_{ys}}{FOS} \)

or \( \frac{33179.5}{d} = \frac{43000}{1} \Rightarrow d = 0.91 \text{ in} \)

Select highest value, \( d = 1.01 \text{ in} \) (Ans)

Equivalent stress method

Equivalent stress in the present case, \( \sigma_{eq} = \sqrt{\sigma_x^2 + 3\tau_{xz}^2} \)

\[
\sigma_{eq,d} = \sqrt{\sigma_{x,d}^2 + 3\tau_{xz,d}^2} = \sqrt{\left(\frac{20362.7}{d^3}\right)^2} = \frac{20362.7}{d^3}
\]

\[
\sigma_{eq,m} = \sqrt{\sigma_{x,m}^2 + 3\tau_{xz,m}^2} = \sqrt{3\times\left(\frac{15128.1}{d^3}\right)^2} = \frac{26195.5}{d^3}
\]

where \( \sigma_{x,d} = \frac{\sigma_{x,max} - \sigma_{x,min}}{2} \), \( \sigma_{x,m} = \frac{\sigma_{x,max} + \sigma_{x,min}}{2} \)

\[
\tau_{xz,d} = \frac{\tau_{xz,max} - \tau_{xz,min}}{2} = 0 \quad \text{and} \quad \tau_{xz,m} = \frac{\tau_{xz,max} + \tau_{xz,min}}{2} = \frac{15128.1}{d^3}
\]

Use of Goodman’s basic criterion, \( \frac{\sigma_{eq,d} \times FOS}{\sigma_{eq}} + \frac{\sigma_{eq,m} \times FOS}{\sigma_{uts}} = 1 \), results in \( d = 1.017 \) in that is nearly identical to the value obtained earlier. Fulfillment of the conditions \( \sigma_{max} < \sigma_{ys} \) and \( \sigma_{eq,d} < \sigma_{ys} \) confirms high cycle fatigue.

3.4. Refer Fig. 4. A hollow tubular steel bar is used as a torsion spring that is subjected to a cyclic pure torque ranging from -5000 in-lb to +15000 in-lb. The wall thickness is desired to be 10% of the outside diameter. The material has ultimate tensile strength of 200000 psi and yield strength of 180000 psi. The fatigue limit is 95000 psi. Find the tube dimensions. Assume factor of safety as unity.

Solution

Given: \( T_{\text{max}} = 15000 \text{ in-lb}, T_{\text{min}} = -5000 \text{ in-lb}, \sigma_{uts} = 200000 \text{ psi}, \sigma_{ys} = 180000 \text{ psi}, \sigma_{e} = 95000 \text{ psi}, FOS = 1 \)

As steel is a ductile material, distortion energy theory is used.

Figure 4 Member of Case 3.4
On Distortion Energy Theory in High Cycle Multi-Axial Fatigue

\[ d_o = d_i + 2 \times \frac{10}{100} \times d_o \text{ or } d_o = \frac{5}{4} d_i \text{ where } d_o \text{ and } d_i \text{ are external and internal diameters of the tube.} \]

Shear stress at outer surface point, \( \tau = \frac{5}{4} \times \frac{16 T}{\pi d_i^3 \left( \left( \frac{d_o}{d_i} \right)^4 - 1 \right)} = 4.41T \)

\[ \tau_{sz,\text{max}} = \frac{4.41 \times 15000}{d_i^3} = 66284.8 \quad \text{and} \quad \tau_{sz,\text{min}} = \frac{-4.41 \times 5000}{d_i^3} = \frac{-22094.9}{d_i^3} \]

Principal stresses are given as follows :-

For positive peak value of \( T \)
\[ \sigma_{1,\text{max}} = \frac{66284.8}{d_i^3}, \sigma_{2,\text{max}} = -\frac{66284.8}{d_i^3} \]

For negative peak value of \( T \)
\[ \sigma_{1,\text{min}} = -\frac{22094.9}{d_i^3}, \sigma_{2,\text{min}} = \frac{22094.9}{d_i^3} \]
\[ \sigma_{1,\text{m}} = \frac{22094.9}{d_i^3}, \sigma_{2,\text{m}} = -\frac{22094.9}{d_i^3} \]
\[ \sigma_{3,\text{max}} = \sigma_{3,\text{min}} = \sigma_{3,\text{m}} = 0 \]

\[ U_{d,\text{max}} = \left[ \frac{1 + \nu}{6E} \right] \left[ (\sigma_{1,\text{max}} - \sigma_{2,\text{max}})^2 + (\sigma_{2,\text{max}} - \sigma_{3,\text{max}})^2 + (\sigma_{3,\text{max}} - \sigma_{1,\text{max}})^2 \right] = \left[ \frac{1 + \nu}{6E} \right] \times \frac{2.62 \times 10^{10}}{d_i^6} \]

Equivalent uni - axial maximum stress, \( \sigma_{\text{max}} = \sqrt{\frac{3E}{1 + \nu} U_{d,\text{max}}} = \frac{114645.8}{d_i^3} \)

\[ U_{d,\text{m}} = \left[ \frac{1 + \nu}{6E} \right] \left[ (\sigma_{1,\text{m}} - \sigma_{2,\text{m}})^2 + (\sigma_{2,\text{m}} - \sigma_{3,\text{m}})^2 + (\sigma_{3,\text{m}} - \sigma_{1,\text{m}})^2 \right] = \left[ \frac{1 + \nu}{6E} \right] \times \frac{2.9 \times 10^9}{d_i^6} \]

Equivalent uni - axial mean stress, \( \sigma_{\text{m}} = \sqrt{\frac{3E}{1 + \nu} U_{d,\text{m}}} = \frac{38269.48}{d_i^3} \)

\[ r = \frac{\sigma_{\text{m}}}{\sigma_{\text{y}}} = \frac{95000}{200000} = 0.475 \]

Modified Goodman’s failure equations are

i) \( \sigma_{\text{max}} - 2\sigma_{\text{m}} \geq \frac{\sigma_{\text{y}}}{\text{FOS}} \)

or \( \frac{114645.8}{d_i^3} - \frac{76538.96}{d_i^3} = 180000 \Rightarrow d_i = 0.59 \text{ in} \)
ii) $\sigma_{\text{max}} - \sigma_{m} \geq \frac{\sigma_{e}}{\text{FOS}}$

\[
\frac{114645.8}{d_{i}^{3}} - \frac{38269.48}{d_{i}^{3}} = \frac{95000}{1} \Rightarrow d_{i} = 0.92 \text{ in}
\]

iii) $\sigma_{\text{max}} - (1 - r)\sigma_{m} \geq \frac{\sigma_{e}}{\text{FOS}}$

\[
\frac{114645.8}{d_{i}^{3}} - 0.525 \times \frac{38269.48}{d_{i}^{3}} = \frac{95000}{1} \Rightarrow d_{i} = 0.99 \text{ in}
\]

iv) $\sigma_{\text{max}} \geq \frac{\sigma_{ys}}{\text{FOS}}$

\[
\frac{114645.8}{d_{i}^{3}} = \frac{180000}{1} \Rightarrow d_{i} = 0.86 \text{ in}
\]

Select highest value, $d_{i} = 0.99 \text{ in}$, $d_{o} = \frac{5}{4} \times d_{i} = 1.24 \text{ in}$ (Ans)

Equivalent stress method

Equivalent stress in the present case

\[
\sigma_{eq} = \sqrt{3\tau_{xz}^2}
\]

\[
\sigma_{eq.a} = \sqrt{3\tau_{xz,a}^2} = \frac{76539}{d_{i}^{3}}
\]

\[
\sigma_{eq.m} = \sqrt{3\tau_{xz.m}^2} = \frac{38269.5}{d_{i}^{3}}
\]

where $\tau_{xz,a} = \frac{\tau_{xz,max} - \tau_{xz,min}}{2} = \frac{44189.85}{d_{i}^{3}}$ and $\tau_{xz,m} = \frac{\tau_{xz,max} + \tau_{xz,min}}{2} = \frac{22094.95}{d_{i}^{3}}$

Use of Goodman’s basic criterion, \[
\frac{\sigma_{eq,a} \times \text{FOS}}{\sigma_{e}} + \frac{\sigma_{eq.m} \times \text{FOS}}{\sigma_{uts}} = 1,
\]

results in $d_{i} = 0.99$ inches that is identical to the value obtained earlier. Fulfillment of the conditions $\sigma_{\text{max}} < \sigma_{ys}$ and $\sigma_{eq.a} < \sigma'_{ys}$ confirms high cycle fatigue.

4. RESULTS

Problem 3.1 deals with a circular bar subjected to, positive-positive, axial and uni-directional cyclic load. Since there are nil shear stresses acting in the given plane, the given plane itself is the principal plane and the principal stresses are equal to the applied stresses. This problem however does not involve multi-axial fatigue and is a precursor to multi-axial problems discussed from 3.2 to 3.4. Problem 3.2 deals with a thin cylinder subjected to, zero-positive, bi-axial, cyclic load in the form of hoop and longitudinal stresses. The given plane is the principal plane due to the absence of shear stresses and the principal stresses are equal to the applied stresses. Problem 3.3 is related to a solid transmission shaft that is under the combined effect of completely reversible bending and constant torsion. Given plane is not the principal plane and the principal stresses differ from the applied stresses. Problem 3.4 discusses a hollow tube that is subjected to negative-positive torsion/pure shear. Given plane is not the principal plane but the principal stresses are equal to the applied shear stresses.
On Distortion Energy Theory in High Cycle Multi-Axial Fatigue

Fulfillment of the conditions $\sigma_{\text{max}} < \sigma_{ys}$ and $\sigma_{eq,a} < \sigma'_{ys}$ in all the problems confirms high cycle fatigue.

First two modified Goodman’s relations for $-ve \ \sigma_m$ i.e. $\sigma_{\text{max}} - 2\sigma_m \geq \frac{\sigma_{ys}}{FOS}$ and $\sigma_{\text{max}} - \sigma_m \geq \frac{\sigma_f}{FOS}$ don’t yield correct results as $\sigma_m$ is a positive quantity in the problems. The third relation $\sigma_{\text{max}} - (1-r)\sigma_m \geq \frac{\sigma_f}{FOS}$ for $+ve \ \sigma_m$ i.e. $0 \leq \sigma_m \leq \frac{\sigma_{ys} - \sigma_f}{1-r}$ is found to be the critical equation in all the problems as it provides the maximum value of the desired parameter. In problem 1, $\sigma_m = \frac{9545}{d^2} = 27420 \text{ psi}$ that is less than $\left( \frac{\sigma_{ys} - \sigma_f}{1-r} \right) = 66666.6 \text{ psi}$. In problem 2, $\sigma_m = \sqrt[3]{\frac{5625}{t}} = 48713.9 \text{ psi}$ that is less than $\left( \frac{\sigma_{ys} - \sigma_f}{1-r} \right) = 166666.6 \text{ psi}$. In problem 3, $\sigma_m = 0$ that is less than $\left( \frac{\sigma_{ys} - \sigma_f}{1-r} \right) = 21568.6 \text{ psi}$. In problem 4, $\sigma_m = \frac{38269.48}{d_i^3} = 39440.9 \text{ psi}$ that is again less than $\left( \frac{\sigma_{ys} - \sigma_f}{1-r} \right) = 161904.76 \text{ psi}$. Since region of mean stress is successfully verified in each problem, the solutions from third relation are valid. The results are substantiated by comparing them with those obtained from a different approach involving equivalent stress and Goodman’s basic criterion. The two are in excellent agreement.

5. CONCLUSIONS

Distortion energy fatigue failure theory meant to investigate multi-axial fatigue in ductile members is reviewed under high cycle conditions. Modified Goodman’s failure relationships and their application in the theory are presented. Some problems are solved to demonstrate the procedure. The results from the theory are substantiated with the help of equivalent stress method and Goodman’s basic criterion.

REFERENCES


