ZAGREB INDICES AND ZAGREB COINDICES OF SOME GRAPH OPERATIONS

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ABSTRACT

A topological index is the graph invariant numerical descriptor calculated from a molecular graph representing a molecule. Classical Zagreb indices and the recently introduced Zagreb coindices are topological indices related to the atom–atom connectivity of the molecular structure represented by the graph $G$. We explore here their basic mathematical properties and present explicit formulae for these new graph invariants under several graph operations.

Key words: Topological Index, Zagreb Indices and Zagreb Coindices, Graph Operations.

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1. INTRODUCTION

A representation of an object giving information only about the number of elements composing it and their connectivity is named as topological representation of an object. A topological representation of a molecule is called molecular graph. A molecular graph is a collection of points representing the atoms in the molecule and set of lines representing the covalent bonds. These points are named vertices and the lines are named edges in graph theory language. The advantage of topological indices is in that they may be used directly as simple numerical descriptors in a comparison with physical, chemical or biological parameters of molecules in Quantitative Structure Property Relationships (QSPR) and in Quantitative Structure Activity Relationships (QSAR).

The first and second Zagreb indices [6] which are dependent on the degrees of adjacent vertices appeared in the topological formula for the $\pi$-electron energy of
conjugated systems, were first introduced by Gutman and Trinajstic (1983). The Zagreb indices are found to have applications in QSPR and QSAR studies. Recently, the first and second Zagreb coindices [1], a new pair of invariants, were introduced in Doslic which are dependent on the degrees of non-adjacent vertices and thereby quantifying a possible influence of remote pairs of vertices to the molecule’s properties. Note that the Zagreb coindices of $G$ are not Zagreb indices of $\overline{G}$: while the defining sums are over the set of edges of $\overline{G}$, the degrees are still with respect to $G$.

2. ZAGREB INDICES AND ZAGRED COINDICES

All graphs considered are finite and simple. Let $G$ be a finite simple graph on $n$ vertices and $m$ edges. We denote the vertex set and the edge set by $V(G)$ and $E(G)$, respectively (Doslic 2010). The complement of $G$, denoted by $\overline{G}$, is a simple graph on the same set of vertices $V(G)$ in which two vertices $u$ and $v$ are adjacent. (i.e) connected by an edge $uv$, iff they are not adjacent in $G$. Hence $uv \in E(\overline{G}) \leftrightarrow uv \notin E(G)$. Obviously, $E(G) \cup E(\overline{G}) = E(K_n)$ and

$$\overline{m} = |E(\overline{G})| = \binom{n}{2} - m.$$ The degree of a vertex $u$ in $G$ is denoted by $d_G(u)$; the degree of the same vertex in $\overline{G}$ is then given by $d_{\overline{G}}(u) = n - 1 - d_G(u).$ The sum of the degrees of vertices of a graph $G$ is twice the number of edges. (i.e) $\sum_{u \in V(G)} d_G(u) = 2m$.

2.1. ZAGREB INDICES

The **First Zagreb Index** of $G$ [5] is denoted by $M_1(G)$ and is defined as

$$M_1(G) = \sum_{u \in V(G)} (d_G(u))^2.$$ The first Zagreb index can also be expressed as a sum over edges in $G$.

$$M_1(G) = \sum_{uv \in E(G)} (d_G(u) + d_G(v))$$

The **Second Zagreb Index** of $G$ is denoted by $M_2(G)$ and is defined as

$$M_2(G) = \sum_{uv \in E(G)} d_G(u)d_G(v)$$
Example 1
Chemical compound: Methylcyclopropane

Here \( v_1, v_2, v_3, v_4 \) represents the labeling of the graph \( G \), while the underlined numbers stand for the valencies of the vertices in \( G \).

\[
M_1(G) = 1^2 + 3^2 + 2^2 + 2^2 = 18
\]
\[
M_2(G) = 1.3 + 3.2 + 3.2 + 2.2 = 19
\]

2.2 Zagreb Coindices
The First Zagreb Coindex [1] of \( G \) is denoted by \( \overline{M}_1(G) \) and is defined as

\[
\overline{M}_1(G) = \sum_{uv \in E(G)} (d_G(u) + d_G(v))
\]

The Second Zagreb Coindex of \( G \) is denoted by \( \overline{M}_2(G) \) and is defined as
\[ \overline{M}_2(G) = \sum_{uv \in E(G)} d_G(u)d_G(v) \]

**EXAMPLE 2**

Chemical compound: Methylcyclopentane

![Figure 2](image)

Here \( E(\overline{G}) = \{v_1v_5, v_1v_6, v_1v_3, v_1v_4, v_2v_5, v_2v_6, v_3v_4, v_3v_6, v_4v_5\} \)

\[ \overline{M}_1(G) = 34 \quad \text{and} \quad \overline{M}_2(G) = 32 \]

**3 BASIC PROPERTIES OF ZAGREB COINDICES**

**Proposition 1**

Let \( G \) be a simple graph \([4]\) on \( n \) vertices and \( m \) edges. Then

\[ \overline{M}_1(\overline{G}) = M_1(G) + 2(n-1)(m-m) \]
Proof

\[ M_1(G) = \sum_{u \in V(G)} d_G(u)^2 \]

We know that

\[ M_1(\overline{G}) = \sum_{u \in V(G)} d_{\overline{G}}(u)^2 = \sum_{u \in V(G)} \left[ n-1-d_G(u) \right]^2 = \sum_{u \in V(G)} (n-1)^2 - 2(n-1) \sum_{u \in V(G)} d_G(u) + \sum_{u \in V(G)} d_G(u)^2 \]

\[ = n(n-1)^2 - 4m(n-1) + M_1(G) \]

\[ M_1(\overline{G}) = 2(n-1)(m-m) + M_1(G) \]

**Proposition 2**

Let \( G \) be a simple graph on \( n \) vertices and \( m \) edges. Then \( \overline{M}_1(G) = 2m(n-1) - M_1(G) \).

Proof

\[ \overline{M}_1(G) = \sum_{uv \in E(G)} (d_G(u) + d_G(v)) \]

The First Zagreb Coindex of \( G \) is

\[ = \sum_{uv \in E(G)} \left[ n-1-d_G(u) + n-1-d_G(v) \right] \]

\[ = \sum_{uv \in E(G)} \left[ 2(n-1)-(d_G(u)+d_G(v)) \right] = 2m(n-1) - M_1(\overline{G}) \]

\[ = 2(n-1) \left\{ \frac{n}{2} - m \right\} - \sum_{u \in V(G)} d_G(u)^2 \]

\[ = 2(n-1) \left\{ \frac{n}{2} - m \right\} - \sum_{u \in V(G)} \left[ (n-1-d_G(u))^2 \right] \]

\[ = 2(n-1) \left\{ \frac{n(n-1)}{2} - m \right\} - \left[ \sum_{u \in V(G)} (n-1)^2 - 2(n-1) \sum_{u \in V(G)} d_G(u) + \sum_{u \in V(G)} d_G(u)^2 \right] \]

\[ = n(n-1)^2 - 2m(n-1) - n(n-1)^2 + 4m(n-1) - M_1(G) \]

\[ \overline{M}_1(G) = 2m(n-1) - M_1(G) \]

**Proposition 3**

Let \( G \) be a simple graph on \( n \) vertices and \( m \) edges. Then \( \overline{M}_2(G) = M_2(\overline{G}) - (n-1)M_1(\overline{G}) + m(n-1)^2 \).
The Second Zagreb Coindex is \( M_2(G) = \sum_{uv \in E(G)} d_G(u)d_G(v) \)

\[
= \sum_{uv \in E(G)} \left( (n-1) - d_G(u) \right) \left( (n-1) - d_G(v) \right)
\]

\[
= \sum_{uv \in E(G)} (n-1)^2 - (n-1) d_G(u) + d_G(v) + \sum_{uv \in E(G)} d_G(u) d_G(v)
\]

\[\overline{M}_2(G) = M_2(\overline{G}) - (n-1)M_1(\overline{G}) + \overline{m}(n-1)^2\]

**Proposition 4**

Let \( G \) be a simple graph on \( n \) vertices and \( m \) edges. Then \( \overline{M}_1(G) = \overline{M}_1(\overline{G}) \).

**Proof**

By applying proposition 2 to \( G \), we get

\[\overline{M}_1(\overline{G}) = 2\overline{m}(n-1) - M_1(G)\]

Now, plugging in the expression for \( M_1(\overline{G}) \) from proposition 1 yields

\[\overline{M}_1(\overline{G}) = 2\overline{m}(n-1) - M_1(G) - 2(n-1)(\overline{m} - m)\]

\[= 2m(n-1) - M_1(G)\]

\[\overline{M}_1(G) = \overline{M}_1(\overline{G}).\]

**Proposition 5**

Let \( G \) be a simple graph on \( n \) vertices and \( m \) edges. Then \( \overline{M}_2(G) = 2m^2 - M_2(G) - \frac{1}{2} M_1(G) \).

**Proof**

The result follows by squaring both sides of the identity \( \sum_{u \in V(G)} d_G(u) = 2m \) and then

Splitting the term \( 2\sum_{uv \in E(G)} d_G(u)d_G(v) \) into two sums, one over the edges of \( G \) and the other over the edges of \( \overline{G} \) (Gutman 2004).

\[
\sum_{uv \in V(G)} d_G(u)^2 + 2\sum_{uv \in E(G)} d_G(u)d_G(u) + 2\sum_{uv \in E(G)} d_G(u)d_G(v) = 4m^2
\]

\[2\sum_{uv \in E(G)} d_G(u)d_G(u) = 4m^2 - \sum_{u \in V(G)} d_G(u)^2 - 2\sum_{uv \in E(G)} d_G(u)d_G(u)\]

\[\sum_{uv \in E(G)} d_G(u)d_G(u) = 2m^2 - \frac{1}{2}\sum_{u \in V(G)} d_G(u)^2 - \sum_{uv \in E(G)} d_G(u)d_G(v)\]
Zagreb Indices and Zagreb Coindices of Some Graph Operations

\[ M_s(G) = 2m^2 - \frac{1}{2}M_1(G) - M_s(G) \]

4. MAIN RESULTS

In this section we introduce the Zagreb indices and coindices of Cartesian product, Disjunction, Composition, Tensor product and Normal product of graphs. All considered operations are binary. Hence, we usually deal with two finite and simple graphs, \( G_1 \) and \( G_2 \). For a given graph \( G_i \), its vertex set and edge set will be denoted by \( V_i \) and \( E_i \) respectively, where \( i = 1, 2 \). When more than two graphs can be combined using a given operation, the values of subscripts will vary accordingly.

4.1 CARTESIAN PRODUCT

For given graphs \( G_1 \) and \( G_2 \), (Ashrafi 2009) Cartesian product of \( G_1 \times G_2 \) is the graph on the vertex sets \( V_1 \times V_2 \) and \( (u_1,v_1) \) and \( (u_2,v_2) \) are adjacent iff \( u_1 = u_2 \) or \( v_1 = v_2 \). The degree of a vertex \( (u_1,v_1) \) of \( G_1 \times G_2 \) is given by \( d_{G_1 \times G_2}(u_1,v_1) = d_{G_1}(u_1) + d_{G_2}(v_1) \).

4.2 DISJUNCTION

The disjunction \( G_1 \lor G_2 \) of two graphs \( G_1 \) and \( G_2 \) is the graph [2] with the vertex sets \( V_1 \times V_2 \) and \( (u_1,v_1) \) and \( (u_2,v_2) \) are adjacent iff \( u_1 \) is adjacent to \( u_2 \) or \( v_1 \) is adjacent to \( v_2 \). The degree of a vertex \( (u_1,v_1) \) of \( G_1 \lor G_2 \) is given by \( d_{G_1 \lor G_2}(u_1,v_1) = n_2d_{G_1}(u_1) + n_1d_{G_2}(v_1) - d_{G_1}(u_1)d_{G_2}(v_1) \).

4.3 COMPOSITION

The composition \( G_1[G_2] \) of two graphs \( G_1 \) and \( G_2 \) is the graph with the vertex sets \( V_1 \times V_2 \) and \( (u_1,v_1) \) and \( (u_2,v_2) \) are adjacent iff \( u_1 \) is adjacent to \( u_2 \) or \( v_1 \) is adjacent to \( v_2 \). The degree of a vertex \( (u_1,v_1) \) of \( G_1[G_2] \) is given by \( d_{G_1[G_2]}(u_1,v_1) = n_2d_{G_1}(u_1) + d_{G_2}(v_1) \).

4.4 TENSOR PRODUCT

Let \( G_1 = (V_1,E_1) \) and \( G_2 = (V_2,E_2) \) be two graphs whose vertex sets are disjoint as also their edge sets (Parthasarathy 1994) [7]. The tensor product \( G = G_1 \wedge G_2 \) of graphs \( G_1 \) and \( G_2 \) has as its vertex set \( V_1 \times V_2 \) and \( (u_1,v_1) \) and \( (u_2,v_2) \) are adjacent iff \( u_1 \) is
adjacent to \( u_2 \) and \( v_1 \) is adjacent to \( v_2 \). The degree of a vertex \((u, v)\) of \( G_1 \land G_2 \) is given by \( d_{G_1 \land G_2}(u, v) = d_{G_1}(u)d_{G_2}(v) \).

### 4.5 NORMAL PRODUCT

The **normal product** \( G = G_1 \odot G_2 \) (strong product or wreath product) of two graphs \( G_1 \) and \( G_2 \) (BaskarBabujee 2012) has as its vertex set \( V_1 \times V_2 \) and \((u_1, v_1)\) and \((u_2, v_2)\) are adjacent iff \( u_1 \) is adjacent to \( u_2 \) and \( v_1 = v_2 \) or \( u_1 = u_2 \) and \( v_1 \) is adjacent to \( v_2 \) or \( u_1 \) is adjacent to \( u_2 \) and \( v_1 \) is adjacent to \( v_2 \). The degree of a vertex \((u, v)\) of \( G_1 \odot G_2 \) is given by \( d_{G_1 \odot G_2}(u, v) = d_{G_1}(u) + d_{G_2}(v) + \left(d_{G_1}(u)d_{G_2}(v)\right) \).

All graphs are finite and simple. We denote the vertex and edge sets of \( G \) by \( V(G) \) and \( E(G) \) respectively. The number of vertices and edges of \( G \) by \( |V(G)| = n \) and \( |E(G)| = m \) respectively. It can be easily seen that

\[
|V(G_1 \times G_2)| = |V(G_1 \lor G_2)| = |V(G_1 \land G_2)|
\]

\[
= |V(G_1 \land G_2)| = |V(G_1 \odot G_2)| = |V(G_1)||V(G_2)|
\]

**LEMMA 6** If \( G_1 \) and \( G_2 \) are two simple graphs then,

\[
|E(G_1 \times G_2)| = |V(G_2)||E(G_1)| + |V(G_1)||E(G_2)|
\]

\[
|E(G_1 \lor G_2)| = |V(G_2)|^2|E(G_1)| + |V(G_1)|^2|E(G_2)| - 2|E(G_1)||E(G_2)|
\]

\[
|E(G_1 \land G_2)| = 2|E(G_1)||E(G_2)|
\]

\[
|E(G_1 \odot G_2)| = 2|E(G_1)||E(G_2)| + |V(G_2)||E(G_1)| + |V(G_1)||E(G_2)|
\]

**PROOF**

The part (a-e) are consequence of definitions and some wellknown results of the book of Imrich and Klavzar [3].

**ZAGREB TYPE INDICES FOR CARTESIAN PRODUCT OF GRAPHS**

**THEOREM 7**

The First Zagreb index of Cartesian product of two graphs \( G_1 \) and \( G_2 \) is

\[
M_1(G_1 \times G_2) = n_2M_1(G_1) + n_1M_1(G_2) + 8m_1m_2
\]
PROOF

\[ M_1(G_1 \times G_2) = \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \left( \sum_{i=1}^{n_1} \left( d_{G_i, G_j}(u_i) + d_{G_i, G_j}(v_j) \right)^2 \right) \]

Using the definition of Cartesian product, we get

\[ = \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \left[ d_{G_i}(u_i) + d_{G_j}(v_j) \right]^2 \]

\[ = \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \left[ d_{G_i}(u_i)^2 + d_{G_j}(v_j)^2 + 2d_{G_i}(u_i)d_{G_j}(v_j) \right] \]

\[ = \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} d_{G_i}(u_i)^2 + \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} d_{G_j}(v_j)^2 + 2\sum_{i=1}^{m_1} d_{G_i}(u_i)\sum_{j=1}^{m_2} d_{G_j}(v_j) \]

\[ = n_2M_1(G_1) + n_1M_1(G_2) + 8m_1m_2 \]

THEOREM 8

The First Zagreb coindex of cartesian product of two graphs \( G_1 \) and \( G_2 \) is

\[ \overline{M}_1(G_1 \times G_2) = 2n_1n_2 \left[ m_1(n_1-1) + m_2(n_2-1) \right] - 8m_1m_2 + n_2\overline{M}_1(G_1) + n_1\overline{M}_2(G_2). \]

PROOF: By Lemma (6-a) and Theorem 7,

Using the property \( \overline{M}_1(G) = 2m(n-1) - M_1(G) \)

\[ \overline{M}_1(G_1 \times G_2) = 2\left| E(G_1 \times G_2) \right| \left( |V(G_1 \times G_2)| - 1 \right) - M_1(G_1 \times G_2) \]

\[ = 2(n_2m_1 + n_1m_2)(n_1n_2-1) - M_1(G_1 \times G_2) \]

\[ = 2(n_1m_2 + n_2m_1)(n_1n_2-1) - 8m_1m_2 - n_2M_1(G_1) - n_1M_1(G_2) \]

\[ = 2(n_1n_2 - 1)(n_1m_1 + n_1m_2) - 8m_1m_2 - n_2\left[ 2m_1(n_1-1) - \overline{M}_1(G_1) \right] - n_1\left[ 2m_2(n_2-1) - \overline{M}_1(G_2) \right] \]

\[ = 2n_1n_2(n_1m_1 + n_1m_2) - 2(n_1m_1 + n_1m_2) - 8m_1m_2 - 2m_1n_2 \\
+2m_2 + n_2\overline{M}_1(G_1) - 2n_2m_2 + 2m_2n_1 + n_1\overline{M}_1(G_2) \]

\[ \overline{M}_1(G_1 \times G_2) = 2n_1n_2 \left[ m_1(n_1-1) + m_2(n_2-1) \right] - 8m_1m_2 + n_2\overline{M}_1(G_1) + n_1\overline{M}_2(G_2). \]
ZAGREB TYPE INDICES FOR DISJUNCTION OF GRAPHS

THEOREM 9

The First Zagreb index of disjunction of two graphs $G_1$ and $G_2$ is

$$M_1(G_1 \lor G_2) = (n_2^2 - 4m_2)n_2M_1(G_1) + (n_1^2 - 4m_1)n_1M_1(G_2) + 8m_1m_2n_1n_2 + M_1(G_1)M_1(G_2)$$

PROOF:

Using the definition of disjunction, we get

$$M_1(G_1 \lor G_2) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} [n_2 d_{G_1}(u_i) + n_2 d_{G_2}(v_j) - (d_{G_1}(u_i)d_{G_2}(v_j))^2]$$

$$= \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \left( n_2 d_{G_1}(u_i) + n_2 d_{G_2}(v_j) \right)^2 + \left( d_{G_1}(u_i)d_{G_2}(v_j) \right)^2$$

$$- 2 \left[ n_2 d_{G_1}(u_i) + n_2 d_{G_2}(v_j) \right] \left[ d_{G_1}(u_i)d_{G_2}(v_j) \right]$$

$$= \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} n_2^2 d_{G_1}(u_i)^2 + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} n_2 d_{G_2}(v_j)^2 + 2n_1 n_2 \sum_{i=1}^{n_1} d_{G_1}(u_i) \sum_{j=1}^{n_2} d_{G_2}(v_j)$$

$$+ \sum_{i=1}^{n_1} d_{G_1}(u_i)^2 \sum_{j=1}^{n_2} d_{G_2}(v_j)^2 - 2n_2 \sum_{i=1}^{n_1} d_{G_1}(u_i)^2 \sum_{j=1}^{n_2} d_{G_2}(v_j) - 2n_1 \sum_{i=1}^{n_1} d_{G_1}(u_i) \sum_{j=1}^{n_2} d_{G_2}(v_j)^2$$

$$= n_2^3 M_1(G_1) + n_1^3 M_1(G_2) + 2n_1 n_2 (2m_1)(2m_2) + M_1(G_1)M_1(G_2)$$

$$- 2n_2 M_1(G_1)(2m_2) + 2n_1(2m_l) M_1(G_2)$$

$$M_1(G_1 \lor G_2) = (n_2^2 - 4m_2)n_2M_1(G_1) + (n_1^2 - 4m_1)n_1M_1(G_2) + 8m_1m_2n_1n_2 + M_1(G_1)M_1(G_2)$$

THEOREM 10

The First Zagreb coindex of disjunction of two graphs $G_1$ and $G_2$ is

$$\overline{M_1}(G_1 \lor G_2) = 2m_1 n_2^2 (n_2 - 1) + 2m_2 n_1^2 (n_1 - 1) - 4m_1m_2(n_1 + n_2)$$

$$+ \left[ n_1^3 - 2m_1 n_2 - 2m_2 \right] \overline{M_1}(G_1) + \left[ n_2^3 - 2m_2 n_1 - 2m_1 \right] \overline{M_1}(G_2) - \overline{M_1}(G_1)\overline{M_1}(G_2)$$

PROOF: By Lemma (6-b) and Theorem 9,

Using the property $\overline{M_1}(G) = 2|E(G)|(\lfloor V(G) \rfloor - 1) - M_1(G)$

$$\overline{M_1}(G) = 2|E(G)|(\lfloor V(G) \rfloor - 1) - M_1(G)$$
The First Zagreb index of composition of two graphs $G_1$ and $G_2$ is

$$M_1(G_1 \circ G_2) = 2|E(G_1 \circ G_2)|(|V(G_1 \circ G_2)| - 1) - M_1(G_1 \circ G_2)$$

$$= 2(n_1^2m_1 + n_2^2m_2 - 2m_1m_2)(n_1n_2 - 1) - M_1(G_1 \circ G_2)$$

$$= 2n_1n_2^3m_1 + 2n_1^3n_2m_2 - 4m_1m_2n_1n_2 - 2m_1^2n_2^2 - 2m_2n_1^2 + 4m_1m_2$$

$$-(n_1^2 - 4m_1)n_2M_1(G_1) - (n_2^2 - 4m_2)n_1M_1(G_2) - 8m_1m_2n_1n_2 - M_1(G_1)M_1(G_2)$$

$$= 4m_1m_2 - 12m_1m_2n_1n_2 + 2n_1^3m_1 + 2n_1^3n_2m_2 - 2n_2^2m_1 - 2n_1^2m_2$$

$$+(4m_1n_2 - n_1^3)[2m_1(n_1 - 1) - M_1(G_1)] + (4m_2n_1 - n_2^3)[2m_2(n_2 - 1) - M_1(G_2)]$$

$$- [2m_1(n_1 - 1) - M_1(G_1)][2m_2(n_2 - 1) - M_1(G_2)]$$

$$M_1(G_1 \circ G_2) = 2n_1n_2^2(n_2 - 1) + 2n_2n_1^2(n_1 - 1) - 4m_1m_2(n_1 + n_2)$$

$$+ [n_1^3 - 2m_2n_2 - 2m_2]M_1(G_1) + [n_2^3 - 2m_1n_1 - 2m_1]M_1(G_2) - M_1(G_1)M_1(G_2)$$

**THEOREM 12**

The First Zagreb coindex of composition of two graphs $G_1$ and $G_2$ is

$$M_1(G_1 \circ G_2) = n_1^3M_1(G_1) + n_2M_1(G_2) + 8n_1m_2$$

**PROOF:**

Using the definition of Composition, we get

$$M_1(G_1 \circ G_2) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} [d_{G_1\circ G_2}(u_i, v_j)]^2$$

$$= \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} [n_2d_{G_1}(u_i) + d_{G_2}(v_j)]^2$$

$$= \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} [n_2^2d_{G_1}(u_i)^2 + d_{G_2}(v_j)^2 + 2n_2d_{G_1}(u_i)d_{G_2}(v_j)]$$

$$= \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} n_2d_{G_1}(u_i)^2 + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} d_{G_2}(v_j)^2 + 2n_2 \sum_{i=1}^{n_1} d_{G_1}(u_i) \sum_{j=1}^{n_2} d_{G_2}(v_j)$$

$$M_1(G_1 \circ G_2) = n_1^3M_1(G_1) + n_2M_1(G_2) + 8n_1m_2$$

**THEOREM 12**

The First Zagreb coindex of composition of two graphs $G_1$ and $G_2$ is

$$M_1(G_1 \circ G_2) = 2n_1n_2m_1(n_1 - 1) + 2n_2^2m_2(n_2 - 1) - 8n_1m_2m_2 + n_2^3M_1(G_1) + n_1M_1(G_2)$$

**PROOF:** By Lemma (6 – c) and Theorem 11,
Using the property \( \overline{M}_1(G) = 2m(n-1) - M_1(G) \)

\[
\overline{M}_1(G) = 2|E(G)||(V(G)|-1) - M_1(G)
\]

\[
\overline{M}_1(G_1[G_2]) = 2|E(G_1[G_2])|(|V(G_1[G_2])| - 1) - M_1(G_1[G_2])
\]

\[
= 2\left(n_1^2m_1 + n_2m_2\right)(n_1n_2 - 1) - M_1(G_1 \times G_2)
\]

\[
= 2n_1n_2(n_1m_2 + n_2^2m_1) - 2(n_1m_2 + n_2^2m_1) - 8n_1n_2m_2 - n_2^3M_1(G_1) - n_1M_1(G_2)
\]

\[
= 2n_1n_2(n_1m_2 + n_2^2m_1) - 2(m_1n_2^2 + n_2m_2) - 8n_1n_2m_2
\]

\[
- n_2^3\left[2m_1(n_1 - 1) - \overline{M}_1(G_1)\right] - n_1\left[2m_2(n_2 - 1) - \overline{M}_1(G_2)\right]
\]

From the above, after simple calculations, the desired results follows

\[
\overline{M}_1(G_1[G_2]) = 2n_1n_2m_2(n_1 - 1) + 2n_2^2m_1(n_2 - 1) - 8n_1n_2m_2 + n_2^3\overline{M}_1(G_1) + n_1\overline{M}_1(G_2)
\]

**ZAGREB TYPE INDICES FOR TENSOR PRODUCT OF GRAPHS**

**THEOREM 13**

If \( G_1 \) and \( G_2 \) are two simple graphs and if \( G_1 \wedge G_2 \) is connected then first Zagreb index of \( G_1 \wedge G_2 \) is \( M_1(G_1 \wedge G_2) = M_1(G_1)M_1(G_2) \).

**PROOF:**

The first Zagreb index of \( G_1 \wedge G_2 \) is

\[
M_1(G_1 \wedge G_2) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \left[ d_{G_1 \wedge G_2}(u_i, v_j) \right]^2
\]

\[
= \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \left[ d_{G_1}(u_i) + d_{G_2}(v_j) \right]^2
\]

\[
= \sum_{i=1}^{n_1} \left( d_{G_1}(u_i) \right)^2 + \sum_{j=1}^{n_2} \left( d_{G_2}(v_j) \right)^2 + \left( \sum_{i=1}^{n_1} \left( d_{G_1}(u_i) \right)^2 \right) M_1(G_2)
\]

\[
M_1(G_1 \wedge G_2) = M_1(G_1)M_1(G_2)
\]

**THEOREM 14**

If \( G_1 \) and \( G_2 \) are two simple graphs and if \( G_1 \wedge G_2 \) is connected then second Zagreb index of \( G_1 \wedge G_2 \) is \( M_2(G_1 \wedge G_2) = 2M_2(G_1)M_2(G_2) \).
Zagreb Indices and Zagreb Coindices of Some Graph Operations

PROOF

The second Zagreb index of $G_1 \wedge G_2$ is

$$M_2(G_1 \wedge G_2) = \sum_{(u, v) \in E(G_1 \wedge G_2)} d_{G_1}(u)d_{G_2}(v)$$

$$= \sum_{(u, v) \in E(G_1 \wedge G_2)} d_{G_1}(u)d_{G_2}(v) + d_{G_1}(v)d_{G_2}(u)$$

$$= 2 \sum_{(u, v) \in E(G_i)} d_{G_i}(u)d_{G_i}(v) + \sum_{(u, v) \in E(G_j)} d_{G_j}(u)d_{G_j}(v)$$

$$M_2(G_1 \wedge G_2) = 2M_2(G_1)M_2(G_2)$$

THEOREM 15

If $G_1$ and $G_2$ are two simple graphs and if $G_1 \wedge G_2$ is connected then first Zagreb coindex of $G_1 \wedge G_2$ is

$$\overline{M}_1(G_1 \wedge G_2) = 4m_1m_2(n_1 + n_2 - 2) + 2m_2(n_2 - 1)\overline{M}_1(G_1)$$

$$+ 2m_1(n_1 - 1)\overline{M}_1(G_2) - \overline{M}_1(G_1)\overline{M}_1(G_2)$$

PROOF: By Lemma (6 – d) and Theorem 13,

We know that

$$\overline{M}_1(G) = 2m(n - 1) - M_1(G)$$

$$\overline{M}_1(G) = 2|E(G)|(|V(G)| - 1) - M_1(G)$$

$$\overline{M}_1(G_1 \wedge G_2) = 2|E(G_1 \wedge G_2)||(|V(G_1 \wedge G_2)| - 1) - M_1(G_1 \wedge G_2)$$

$$= 2 \left[ 2|E(G_1)||E(G_2)| \left( |V(G_1)| - 1 \right) - M_1(G_1)M_1(G_2) \right]$$

$$= 4m_1m_2(n_1n_2 - 1) - M_1(G_1)M_1(G_2)$$

$$= 4m_1m_2(n_1n_2 - 1) - \overline{M}_1(G_1)\overline{M}_1(G_2)$$

$$= 4m_1m_2(n_1n_2 - 1) - \overline{M}_1(G_1)\overline{M}_1(G_2)$$

$$= 4m_1m_2(n_1n_2 - 1) - 4m_1m_2(n_1 - 1)(n_2 - 1)$$

$$+ 2m_1(n_1 - 1)\overline{M}_1(G_2) + 2m_2(n_2 - 1) - \overline{M}_1(G_1) - \overline{M}_1(G_1)\overline{M}_1(G_2)$$

$$= 4m_1m_2(n_1n_2 - 1 - n_1n_2 + n_1 + n_2 - 1)$$

$$+ 2m_1(n_1 - 1)\overline{M}_1(G_2) + 2m_2(n_2 - 1) - \overline{M}_1(G_1) - \overline{M}_1(G_1)\overline{M}_1(G_2)$$

$$\overline{M}_1(G_1 \wedge G_2) = 4m_1m_2(n_1 + n_2 - 2) + 2m_2(n_2 - 1)\overline{M}_1(G_1)$$

$$+ 2m_1(n_1 - 1)\overline{M}_1(G_2) - \overline{M}_1(G_1)\overline{M}_1(G_2)$$
THEOREM 16
Let $G_1$ and $G_2$ be two simple graphs then, $G_1 \wedge G_2$ is connected such that second Zagreb coindex of $G_1 \wedge G_2$ is
\[
M_2(G_1 \wedge G_2) = 2m_2M_2(G_1)(2m_2 - n_2 + 1) + 2m_1M_2(G_2)(2m_1 - n_1 + 1) \\
+ 2m_2M_1(G_1)(n_2 - m_2 - 1) + 2m_1M_1(G_2)(n_1 - m_1 - 1) - 2M_2(G_1)M_2(G_2) \\
+ 4m_2(m_1(n_2 - 1) + m_2(n_1 - 1) - (n_1 - 1)(n_2 - 1)) + M_1(G_2)M_2(G_1) \\
+ M_1(G_1)M_2(G_2) - M_1(G_1)M_1(G_2)
\]
PROOF: By Lemma (6 – d), Theorem 13 and Theorem 14,
We know that
\[
M_2(G) = 2m^2 - M_2(G) - \frac{1}{2}M_1(G)
\]
We have that
\[
M_2(G_1 \wedge G_2) = 2|E(G_1 \wedge G_2)|^2 - M_2(G_1 \wedge G_2) - \frac{1}{2}M_1(G_1 \wedge G_2)
\]
\[
= 2[2|E(G_1)||E(G_2)|]^2 - 2M_2(G_1)M_2(G_2) - \frac{1}{2}M_1(G_1)M_1(G_2)
\]
\[
= 2(2m_1m_2)^2 - 2M_2(G_1)M_2(G_2) - \frac{1}{2}M_1(G_1)M_1(G_2)
\]
\[
= 8m_1^2m_2^2 - 2\left[2m_1^2 - M_2(G_1) - \frac{1}{2}M_1(G_1)\right]\left[2m_2^2 - M_2(G_2) - \frac{1}{2}M_1(G_2)\right]
\]
\[
- \frac{1}{2}\left[2m_1(n_1 - 1) - M_1(G_1)\right]\left[2m_2(n_2 - 1) - M_1(G_2)\right]
\]
\[
= 8m_1^2m_2^2 - 2\left[2m_1^2 - M_2(G_1) - \frac{1}{2}\left(2m_1(n_1 - 1) - M_1(G_1)\right)\right]
\]
\[
\left[2m_2^2 - M_2(G_2) - \frac{1}{2}\left(2m_2(n_2 - 1) - M_1(G_2)\right)\right]
\]
\[
- \frac{1}{2}\left[2m_1(n_1 - 1) - M_1(G_1)\right]\left[2m_2(n_2 - 1) - M_1(G_2)\right]
\]
By above calculations, we conclude that
\[
M_2(G_1 \wedge G_2) = 2m_2M_2(G_1)(2m_2 - n_2 + 1) + 2m_1M_2(G_2)(2m_1 - n_1 + 1) \\
+ 2m_2M_1(G_1)(n_2 - m_2 - 1) + 2m_1M_1(G_2)(n_1 - m_1 - 1) - 2M_2(G_1)M_2(G_2)
\]
Zagreb Indices and Zagreb Coinindices of Some Graph Operations

\[+4m_1m_2(n_1(n_2 - 1) + m_2(n_1 - 1) - (n_1 - 1)(n_2 - 1)) + M_1(G_1)M_2(G_1)
+ M_1(G_1)M_2(G_2) - M_1(G_1)M_1(G_2)\]

ZAGREB TYPE INDICES FOR NORMAL PRODUCT OF GRAPHS

THEOREM 17

The First Zagreb index of normal product of two graphs \(G_1\) and \(G_2\) is

\[M_1(G_1 \circ G_2) = (4m_1 + n_1)M_1(G_2) + (4m_2 + n_2)M_1(G_1) + 8m_1m_2 + M_1(G_1)M_1(G_2)\]

PROOF

\[M_1(G_1 \circ G_2) = \sum_{i=1}^{m_1} \sum_{j=1}^{n_1} \left[ d_{G_1}(u_i) + d_{G_1}(v_j) + \left( d_{G_1}(u_i)d_{G_2}(v_j) \right) \right]^2\]

\[= \sum_{i=1}^{m_1} \sum_{j=1}^{n_1} \left[ d_{G_1}(u_i)^2 + d_{G_1}(v_j)^2 + 2d_{G_1}(u_i)d_{G_1}(v_j) \right] + d_{G_1}(u_i)^2d_{G_1}(v_j)^2 + 2d_{G_1}(u_i)^2d_{G_2}(v_j) + 2d_{G_1}(u_i)d_{G_2}(v_j)^2\]

\[= \sum_{i=1}^{m_1} \sum_{j=1}^{n_1} d_{G_1}(u_i)^2 + \sum_{i=1}^{m_1} \sum_{j=1}^{n_1} d_{G_2}(v_j)^2 + 2\sum_{i=1}^{m_1} d_{G_1}(u_i)\sum_{j=1}^{n_1} d_{G_2}(v_j)\]

\[+ \sum_{j=1}^{n_1} d_{G_1}(u_i)^2\sum_{j=1}^{n_1} d_{G_2}(v_j)^2 + 2\sum_{j=1}^{n_1} d_{G_1}(u_i)^2\sum_{j=1}^{n_1} d_{G_2}(v_j) + 2\sum_{j=1}^{n_1} d_{G_1}(u_i)\sum_{j=1}^{n_1} d_{G_2}(v_j)^2\]

\[= n_2M_1(G_1) + n_1M_1(G_2) + 2(2m_1)(2m_2) + M_1(G_1)M_1(G_2)\]

\[+ 2M_1(G_1)(2m_2) + 2(2m_1)M_1(G_2)\]

\[= n_2M_1(G_1) + n_1M_1(G_2) + 8m_1m_2 + M_1(G_1)M_1(G_2) + 4m_2M_1(G_1) + 4m_1M_1(G_2)\]

\[M_1(G_1 \circ G_2) = (4m_1 + n_1)M_1(G_2) + (4m_2 + n_2)M_1(G_1) + 8m_1m_2 + M_1(G_1)M_1(G_2)\]

THEOREM 18

The First Zagreb coindex of normal product of two graphs \(G_1\) and \(G_2\) is

\[\overline{M_1(G_1 \circ G_2)} = 2n_1n_2m_1(n_2 - 1) + 2n_2n_1m_2(n_1 - 1) - 4m_1m_2(n_1 + n_2) + (4m_2 + n_2)M_1(G_1)\]

\[\overline{M_1(G_1)M_1(G_2)} + M_1(G_1)(n_2 + 2m_2 + 2m_2) + M_1(G_2)(n_1 + 2m_1 + 2m_1)\]
PROOF: By Lemma (6 – e) and Theorem 17,

Using the property  \( \overline{M}_1(G) = 2m(n-1) - M_1(G) \)

\( \overline{M}_1(G) = 2|E(G)||V(G)| - 1 - M_1(G) \)

\( \overline{M}_1(G_1 \circ G_2) = 2|E(G_1 \circ G_2)||V(G_1 \circ G_2)| - 1 - M_1(G_1 \circ G_2) \)

\( = 2(2m_1m_2 + n_1m_2 + n_2m_1)(n_1n_2 - 1) - M_1(G_1 \circ G_2) \)

\( \overline{M}_1(G_1 \circ G_2) = 4m_1m_2n_1n_2 - 4m_1m_2 + 2n_1^2n_2m_2 - 2n_1m_2 + 2n_2^2m_1 - 2n_2m_1 \)

\(-[(4m_1 + n_1)M_1(G_2) + [(4m_2 + n_2)M_1(G_1) + 8m_1m_2 + M_1(G_1)M_1(G_2)] \]

\( = 4m_1m_2n_1n_2 - 4m_1m_2 + 2n_1^2n_2m_2 - 2n_1m_2 + 2n_2^2m_1 - 2n_2m_1 \)

\(-[(4m_1 + n_1)[2m_2(n_2 - 1) - \overline{M}_1(G_2)] - (4m_2 + n_2)[2m_1(n_1 - 1) - \overline{M}_1(G_1)] \]

\(-8m_1m_2 - [(2m_1(n_1 - 1) - \overline{M}_1(G_1)) - (2m_2(n_2 - 1) - \overline{M}_1(G_2)) \]

\( = 4m_1m_2n_1n_2 - 4m_1m_2 + 2n_1^2n_2m_2 - 2n_1m_2 + 2n_2^2m_1 - 2n_2m_1 - 8m_1m_2(n_2 - 1) \)

\(+4m_1\overline{M}_1(G_2) - 2n_1m_2(n_2 - 1) + n_1\overline{M}_1(G_2) - 8m_1m_2(n_1 - 1) + 4m_2\overline{M}_1(G_1) \)

\(-2n_2m_1(n_1 - 1) + n_2\overline{M}_1(G_1) - 8m_2m_2 - 4m_2m_2(n_1 - 1)(n_2 - 1) \)

\(+2m_2(n_2 - 1)\overline{M}_1(G_1) - \overline{M}_1(G_1)\overline{M}_1(G_2) \)

\(\overline{M}_1(G_1 \circ G_2) = 2n_1m_2m_1(n_2 - 1) + 2n_1n_2m_2(n_1 - 1) - 4m_1m_2(n_1 + n_2) + (4m_2 + n_2)M_1(G_1) \)

\(-\overline{M}_1(G_1)\overline{M}_1(G_2) + \overline{M}_1(G_1)(n_2 + 2m_2n_2 + 2m_2) + \overline{M}_1(G_2)(n_1 + 2m_1n_1 + 2m_1) \)

CONCLUSION

In this paper, we have studied the Zagreb indices and Zagreb coindices and also we have investigated their basic mathematical properties and obtained explicit formulae for computing their values under several graph operations namely Cartesian product, Disjunction, Composition, Tensor product and Normal product of graphs. Nevertheless, there are still many other graph operations and special classes of graphs that are not covered here and we mention some possible directions for future research.
REFERENCES


