DECOMPOSITION OF COMPLETE GRAPHS INTO CIRCULANT GRAPHS 
AND ITS APPLICATION

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ABSTRACT

In this paper, the decomposition of complete graphs $K_{2m+1}$ for $m \geq 2$ into circulant graphs has been discussed. Two theorems have been established for different values of $m \geq 2$ relating to $2m+1$ is prime, $2m+1 = 3 \cdot 3^n$ for $n \geq 1$ and $2m+1 = 3s$ for $s \geq 5$ is also prime. Finally, an algorithm under different situation for the traveling salesman problem have been discussed when the weights of edges are non-repeated of the complete graph $K_{2m+1}$ for $m \geq 2$.

Keywords: Algorithm, Circulant Graphs, Complete Graphs, Hamiltonian Graphs, Traveling Salesman Problem (TSP).

1. INTRODUCTION

Circulant graph have many applications in areas like telecommunication network, VLSI design and distributed computing. Circulant graph is a natural extension of a ring, with increased connectivity. Zbgniew R. Bogdanowicz [1] give the necessary and sufficient conditions under which a directed circulant graph $G$ of order $n$ and with $k$ jumps can be decomposed into $k$ pair wise arc-disjoint anti-directed Hamiltonian cycles, each induced by two jumps. In addition, the necessary condition for complete decomposition of $G$ into arbitrary anti-directed Hamiltonian cycles has been discussed. Matthew Dean [2] shown that there is a Hamiltonian cycle decomposition of every 6-regular circulant graph $\langle S \rangle_n$ in which $S$ has an element of order $n$, $S \subseteq Z_n \setminus \{0\}$ has vertex set $Z_n$ and edge set $\{[x, x+s] | x \in Z_n, s \in S\}$. S. El-Zanati, Kyle King and Jeff Mudrock [3] discussed the cyclic decomposition of circulant graphs into almost bipartite graphs. Dalibor Froncek [4] used...
labeling to show the cyclic decomposition of complete graphs into $K_{n,n} + e$: the missing case. J.C. Bermond, O. Favaron and M. Maheo [7] prove that any 4-regular connected Cayley graph on a finite abelian group can be decomposed into two Hamiltonian cycles. Daniel K. Biss [5] shown that the circulant graph $G(cd^m,d)$ is Hamiltonian decomposable for all positive integers $c,d$ and $m$ with $c < d$ where the graph $G(N,d)$ has vertex set $V = \{0,1,2,\ldots,N-1\}$, with $\{v,w\}$ an edge if $v - w \equiv \pm d (\text{mod } N)$ for some $0 \leq i \leq \lceil \log_d N \rceil - 1$. This extends work of Michenean [6]. Marsha F. Foregger [8] settles Kotzig’s [9] conjecture that the Cartesian product of any three cycles can be decomposed into three Hamiltonian cycles. He also show that the Cartesian product of $2^a3^b$ graphs, each decomposed into $2^a3^b$m Hamiltonian cycles. Dave Morris [10] discussed the status of the search for Hamiltonian cycles in circulant graphs and circulant digraphs. R. Anitha and R. S. Lekshmi [13] have proved that the complete graph $K_{2n}$ can be decomposed into $n - 2$ n-suns, a Hamiltonian cycle and a perfect matching, when $n$ is even and for odd cases, the decomposition is $n - 1$ n-suns and a perfect matching. They also discussed that a spanning tree decomposition of even order complete graph using the labeling scheme of $n$-suns decomposition. A complete bipartite graph $K_{n,n}$ can be decomposed into $\frac{n}{2}$ n-suns when $\frac{n}{2}$ is even. When $\frac{n}{2}$ is odd, $K_{n,n}$ can be decomposed into $\frac{(n - 2)}{2}$ n-suns and a Hamiltonian circuit [13]. Tay-Woei Shyu [14] investigated the decomposition of complete graph $K_n$ cycles $C_r$’s and stars $S_k$’s and studied necessary or sufficient conditions for such a decomposition to exist. Dalibor Froncek [15] show that every bipartite graph $H$ which decomposes $K_k$ and $K_n$ also decomposes $K_{kn}$. Darryn E. Bryant and Peter Adams [16] discussed that for all $v \geq 3$, the complete graph $K_v$ on $v$ vertices can be decomposed into $v - 2$ edge disjoint cycles whose lengths are $3,3,4,5,\ldots,v - 1$ and for all $v \geq 7$, $K_v$ can be decomposed into $v - 3$ edge disjoint cycles whose lengths are $3,4,\ldots,v - 4,v - 2,v - 1,v$. Fu, Hwang, Jimbo, Mutoh and Shiue [17] considered the problem of decomposing a complete graph into the Cartesian product of two complete graphs $K_r$ and $K_c$ and they found a general method of constructing such decomposition using various sorts of combinatorial designs. The traveling salesman problem (TSP) in the circulant weighted undirected graph case have been discussed by I. Gerace and F. Greco [11]. They discussed an upper bound and a lower bound for the Hamiltonian case and analyzed the two stripe case. Moreover, I. Gerace and F. Greco [12] studied short SCTSP (symmetric circulant traveling salesman problem) and they presented an upper bound, a lower bound and a polynomial time 2-approximation algorithm for the general case of SCTSP (symmetric circulant traveling salesman problem). Further, different types of algorithms and results have also been focused by B.Kalita [19-22], J.K. choudhury [23-24] and A. Dutta[25-26].

In this paper, we present a theorem for decomposition of complete graphs $K_{2m+1}$ into $m$ edge disjoint circulant graphs $C_{2mb+1}(j)$ for $m \geq 2$ where $j$ is the jump, $1 \leq j \leq m$. Thereafter, we present a theorem for the decomposition of $K_{2m+1}$ into edge disjoint circulant graphs whenever $2m+1$ is a prime, $2m+1=3 \cdot 3^s$, $n \geq 1$ and $2m+1=3s$, $s \geq 5$ is a prime and this circulant graphs lead to determine edge disjoint Hamiltonian cycle. Finally, an algorithm has been developed to determine the least cost route of a traveler.
The paper is organized as follows
The section 1 includes the introduction part containing works of other researcher. Section 2 includes notations and terminologies. In section 3, two theorems are stated and proved. Section 4 includes an algorithm. Section 5 explains experimental result and the conclusion is included in section 6.

2. NOTATION AND TERMINOLOGY

The notation and terminologies have been considered from the standard references [1-29]

Definition 2.1[18] (circulant matrix): Every \( n \times n \) matrix \( C \) of the form

\[
C = \begin{pmatrix}
c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\
c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\
\vdots & c_{n-1} & c_0 & \ddots & \vdots \\
c_2 & \ddots & \ddots & \ddots & c_1 \\
c_1 & c_2 & \cdots & c_{n-1} & c_0
\end{pmatrix}
\]

is called a circulant matrix. The matrix \( C \) is completely determined by its first row because other rows are rotations of the first row. \( C \) is symmetric if \( c_{n-i} = c_i \) for \( i = 1,2,\ldots, n-1 \). Further, \( C \) is an adjacency matrix if \( c_0 = 0 \) and \( c_{n-1} = c_i \in \{0, 1\} \).

Definition 2.2[18] (circulant graph): A circulant graph is a graph which has a circulant adjacency matrix.

Examples of circulant graphs are the cycle \( C_n \), the complete graph \( K_n \), and the complete bipartite graph \( K_{n,n} \).

Circulant Graph 2.3 [27]: For a given positive integer, let \( n_1, n_2, \ldots, n_k \) be a sequence of integers where

\[0 < n_1 < n_2 < \ldots < n_k < \frac{(p+1)}{2} \]

Then the circulant graph \( C_p(n_1, n_2, \ldots, n_k) \) is the graph on \( p \) nodes \( v_1, v_2, \ldots, v_p \) with vertex \( v_i \) adjacent to each vertex \( v_{i \pm n_j (\mod p)} \). The values \( n_i \) are called jump sizes.

Definition 2.4[28]: A graph \( G \) is said to be decomposable into the subgraphs \( H_1, H_2, H_3, \ldots, H_k, 1 \leq i \leq k \), having no isolated vertices and \( \{E(H_1), E(H_2), E(H_3), \ldots, E(H_k)\} \) is a partition of \( E(G) \).

3. THEOREMS

3.1 Theorem: The complete graph \( K_{2m+1} \) for \( m \geq 2 \) can be decomposed to \( m \) edge disjoint circulant graph \( C_{2m+1}(j) \) where \( j \) is the jump and \( 1 \leq j \leq m \).
Proof: Consider the complete graph $K_5 = K_{2\times 2+1}$. Then vertex set of $K_5$ is $V = \{v_1, v_2, v_3, v_4, v_5\}$ and it has $\frac{n(n-1)}{2} = \frac{5\times 4}{2} = 10$ edges. Let us partitioned the edges of $K_5$ into 2 distinct sets containing 5 edges each as $E_1 = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1\}$ and $E_2 = \{v_1v_3, v_2v_4, v_3v_5, v_4v_1, v_5v_2\}$ respectively. Adjoining edges of $E_1$ and $E_2$, we can construct 2 edge disjoint circulant graphs $C_5(1)$ and $C_5(2)$ for the jumps $j = 1$ and $j = 2$ respectively and their respective permutations can be represented as $f_1$ and $f_2$ where

\[ f_1 = \begin{pmatrix}
    v_1 & v_2 & v_3 & v_4 & v_5 \\
    v_2 & v_3 & v_4 & v_5 & v_1
\end{pmatrix} \quad \text{and} \quad f_2 = \begin{pmatrix}
    v_1 & v_2 & v_3 & v_4 & v_5 \\
    v_3 & v_4 & v_5 & v_1 & v_2
\end{pmatrix} \]

i.e., $f_1 = (v_1v_2v_3v_4v_5)$ and $f_2 = (v_1v_3v_2v_4v_5)$. Figure-1 shows two circulant graphs $C_5(1)$ and $C_5(2)$ for the complete graph $K_5$.

Now, we append the edges to the circulant graphs as

\[ f_j(v_i) = \begin{cases}
    v_{i+j} & \text{if } 1 \leq i \leq 4, j = 1 \\
    v_{i-4} & \text{if } i = 5
\end{cases} \]

and $f_j(v_i) = \begin{cases}
    v_{i+j} & \text{if } 1 \leq i \leq 3, j = 2 \\
    v_{i-3} & \text{if } 4 \leq i \leq 5
\end{cases} \]

Similarly, consider the complete graph $K_7 = K_{2\times 3+1}$ whose vertex set is $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and have $\frac{n(n-1)}{2} = \frac{7\times 6}{2} = 21$ edges. Let us partitioned the edges of $K_7$ into 3 distinct sets consisting of 7 edges each, as $E_1 = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_7, v_7v_1\}$, $E_2 = \{v_1v_3, v_2v_4, v_3v_5, v_4v_6, v_5v_7, v_6v_1, v_7v_2\}$ and $E_3 = \{v_1v_4, v_2v_5, v_3v_6, v_4v_7, v_5v_1, v_6v_2, v_7v_3\}$. Adjoining 7 edges each of $E_1$, $E_2$ and $E_3$, we can construct 3 edge disjoint circulant graphs $C_7(1)$, $C_7(2)$ and $C_7(3)$ for the jumps $j = 1$, $j = 2$ and $j = 3$ respectively and their respective permutations are $f_1$, $f_2$ and $f_3$ where
\[ f_1 = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\ v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_1 \end{pmatrix}, \quad f_2 = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\ v_3 & v_4 & v_5 & v_6 & v_7 & v_1 & v_2 \end{pmatrix} \]

and

\[ f_3 = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\ v_4 & v_5 & v_6 & v_7 & v_1 & v_2 & v_3 \end{pmatrix} \]

i.e., \( f_1 = (v_1v_2v_3v_4v_5v_6v_7) \), \( f_2 = (v_1v_2v_3v_4v_5v_6v_7) \) and \( f_3 = (v_1v_2v_3v_4v_5v_6v_7) \). Figure-2 shows circulant graphs \( C_7(1), C_7(2) \) and \( C_7(3) \).

The edges to the circulant graphs can be appended as

\[
\begin{align*}
  f_j(v_i) &= \begin{cases} 
  v_{i+j} & \text{if } 1 \leq i \leq 6 \\
  v_{i-6} & \text{if } i = 7 
  \end{cases}, & j &= 1 \\
  f_j(v_i) &= \begin{cases} 
  v_{i+j} & \text{if } 1 \leq i \leq 5 \\
  v_{i-5} & \text{if } 6 \leq i \leq 7 
  \end{cases}, & j &= 2 \\
  \text{and} \\
  f_j(v_i) &= \begin{cases} 
  v_{i+j} & \text{if } 1 \leq i \leq 4 \\
  v_{i-4} & \text{if } 5 \leq i \leq 7 
  \end{cases}, & j &= 3
\end{align*}
\]

Similarly, partitioning the edges of \( K_9 \) into 4 distinct sets containing 9 edges each as

\[
\begin{align*}
  E_1 &= \{ v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_7, v_7v_8, v_8v_9, v_9v_1 \}, \\
  E_2 &= \{ v_1v_3, v_2v_4, v_3v_5, v_4v_6, v_5v_7, v_6v_8, v_7v_9, v_8v_1, v_9v_2 \}, \\
  E_3 &= \{ v_1v_4, v_2v_5, v_3v_6, v_4v_7, v_5v_8, v_6v_9, v_7v_1, v_8v_2, v_9v_3 \}, \\
  \text{and} \\
  E_4 &= \{ v_1v_5, v_2v_6, v_3v_7, v_4v_8, v_5v_9, v_6v_1, v_7v_2, v_8v_3, v_9v_4 \}
\end{align*}
\]

respectively, we can construct 4 edge disjoint circulant graphs \( C_9(1), C_9(2), C_9(3) \) and \( C_9(4) \) for the jumps \( j = 1, j = 2, j = 3 \) and \( j = 4 \) respectively and their corresponding permutations are given by
\[ f_1 = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 \\ v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_1 \end{pmatrix} = \left( v_1 v_2 v_3 v_4 v_5 v_8 v_7 v_6 v_9 \right), \]

\[ f_2 = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 \\ v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_1 & v_2 \end{pmatrix} = \left( v_1 v_3 v_5 v_7 v_9 v_2 v_4 v_6 v_8 \right), \]

\[ f_3 = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 \\ v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_1 & v_2 & v_3 \end{pmatrix} = \left( v_1 v_4 v_7 \right) \left( v_2 v_5 v_8 \right) \left( v_3 v_6 v_9 \right) \]

and 
\[ f_4 = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 \\ v_5 & v_6 & v_7 & v_8 & v_9 & v_1 & v_2 & v_3 & v_4 \end{pmatrix} = \left( v_1 v_5 v_9 v_8 v_4 v_3 v_7 v_2 v_6 \right) \]

respectively, whose structures are shown in Figure-3.

![Figure-3](image_url)

Append the edges to the circulant graphs as

\[ f_j(v_i) = \begin{cases} v_{i+j} & \text{if } 1 \leq i \leq 8 \\ v_{i-8} & \text{if } i = 9 \end{cases}, \quad j = 1 \]

\[ f_j(v_i) = \begin{cases} v_{i+j} & \text{if } 1 \leq i \leq 7 \\ v_{i-7} & \text{if } 8 \leq i \leq 9 \end{cases}, \quad j = 2 \]

\[ f_j(v_i) = \begin{cases} v_{i+j} & \text{if } 1 \leq i \leq 6 \\ v_{i-6} & \text{if } 7 \leq i \leq 9 \end{cases}, \quad j = 3 \]

and 
\[ f_j(v_i) = \begin{cases} v_{i+j} & \text{if } 1 \leq i \leq 5 \\ v_{i-5} & \text{if } 6 \leq i \leq 9 \end{cases}, \quad j = 4 \]
From the above discussion, it is seen that the complete graph \( K_{2m+1} \) for \( m \geq 2 \) can be decomposed into \( m \) edge disjoint circulant graph \( C_{2m+1}(j) \). Here \( j \) represents the jump of the circulant also depends upon \( m \). If \( m = 2 \), then \( j = 1, 2 \) and we have 2 circulant graphs \( C_5(1) \) and \( C_5(2) \).

Again, if \( m = 3 \), then \( j = 1, 2, 3 \) and then \( K_7 \) decomposes into 3 edge disjoint circulant graphs \( C_7(1), C_7(2) \) and \( C_7(3) \) and so on. Moreover, in their respective permutations, the images of first element and last element in the first row are respectively \( v_{j+1} \) and \( v_j \). Here we excluded the identity permutations as well as permutations which produce the same circulant graphs, i.e., \( \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 \\ v_2 & v_3 & v_4 & v_5 & v_1 \end{pmatrix} \) and \( \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 \\ v_5 & v_1 & v_2 & v_3 & v_4 \end{pmatrix} \) represents the same circulant graph \( C_5(1) \). Therefore, as discussed above, we get 2 permutations if \( m = 2 \), 3 permutations if \( m = 3 \) etc, i.e., number of permutations are depends upon the value of \( m \geq 2 \).

Finally, we can comment that the complete graph \( K_{2m+1} \) for \( m \geq 2 \) can be decomposed into \( m \) edge disjoint circulant graph \( C_{2m+1}(j) \) for the jump \( 1 \leq j \leq m \) and their corresponding \( m \) permutations are defined as

\[
f_j(v_i) = \begin{cases} 
   v_{i+j} & \text{if } 1 \leq i \leq 2m-(j-1) \\
   v_{i+(j-1)-2m} & \text{if } 2m-(j-2) \leq i \leq 2m+1
\end{cases}
\]

for \( 1 \leq j \leq m \) and which complete the theorem.

Note: Here, complete graphs \( K_{2m+1} \) for \( m \geq 2 \) has odd number of vertices and therefore they play different role for different values of \( m \geq 2 \). In the next theorem we discuss only three cases.

3.2 Theorem: Let \( K_{2m+1} \) be a complete graph for \( m \geq 2 \). Then following conditions are satisfied

(a) If \( p = 2m+1 \) is a prime number, then \( K_p \) can be decomposed into \( m \) edge disjoint circulant graphs \( C_p(j) \) where \( j \) is the jump and \( 1 \leq j \leq m \).

(b) If \( q = 2m+1 = 3 \cdot 3^s \) where \( m = \frac{1}{2}(3^{s+1} - 1) \) for \( n \geq 1 \), then \( K_q \) can be decomposed into

(i) \( 3^n \) edge disjoint circulant graphs \( C_q(j) \) for the jump \( 1 \leq j \leq m \) but \( j \neq 3l \) for \( 1 \leq l \leq \frac{1}{2}(3^n - 1) \) and

(ii) a circulant graph \( C_q(3,6,9,\ldots,3l) \) for the jump \( 3l \) where \( 1 \leq l \leq \frac{1}{2}(3^n - 1) \).

(c) If \( r = 2m+1 = 3s \) where \( s \geq 5 \) is a prime number, then \( K_r \) can be decomposed into

(i) \( s-1 \) edge disjoint circulant graphs \( C_r(j) \) for the jump \( 1 \leq j \leq m \) where \( j \neq s \) and \( j \neq 3e \), \( 1 \leq e \leq \frac{1}{3}(m-1) \)

(ii) a circulant graph \( C_r(3,6,9,\ldots,3e) \) for the jump \( 1 \leq e \leq \frac{1}{3}(m-1) \) and

(iii) a circulant graph \( C_r(s) \) for the jump \( s \).
Proof: (a) When \( p = 2m + 1 \) is prime, then it follows that the complete graph \( K_5, K_7, K_{11}, K_{13}, \ldots \) for \( m = 2, 3, 5, 6, \ldots \) exist and the proof is same as discussed in Theorem 3.1. Hence the complete graph \( K_p \) where \( p = 2m + 1 \) is a prime for \( m \geq 2 \) can be decomposed into \( m = \frac{1}{2}(p - 1) \) edge disjoint circulant graphs \( C_p(j) \) for the jump \( 1 \leq j \leq m \).

In the above discussion, we observed that each edge disjoint circulant graph \( C_p(j) \) for the jump \( 1 \leq j \leq \frac{1}{2}(p - 1) \) is regular of degree two. In other words, they are cycle \( pC \) of length \( p \) and therefore they represent edge disjoint Hamiltonian circuits. So in this discussion, number of edge disjoint Hamiltonian circuits in the complete graph \( K_p \) is \( m \) if \( p = 2m + 1 \) is a prime for \( m \geq 2 \).

Proof: (b) Let \( K_{2m+1} \) be a complete graph for \( m \geq 2 \). Suppose that \( q = 2m + 1 = 3 \cdot 3^n \) such that \( m = \frac{1}{2}(3^n - 1) \) for \( n \geq 1 \). Then \( V = \{v_1, v_2, v_3, \ldots, v_q\} \) is the vertex set of \( K_q \) and it has \( \frac{q(q - 1)}{2} \) edges.

Let us define \( f_j: V(K_q) \rightarrow V(K_q) \) as follows:

(i) when \( 1 \leq j \leq \frac{1}{2}(3^n - 1) \) but \( j \neq 3l \) for \( 1 \leq l \leq \frac{1}{2}(3^n - 1) \) \( \forall n \geq 1 \)

\[
f_j(v) = \begin{cases} v_{i+j} & \text{if } 1 \leq i \leq q - j \\ v_{i+j-q} & \text{if } q - j + 1 \leq i \leq q \\ \end{cases}
\]

and (ii) when \( 1 \leq l \leq \frac{1}{2}(3^n - 1) \) \( \forall n \geq 1 \), \( f_j(v_i) = \begin{cases} v_{i+3l} & \text{if } 1 \leq i \leq q - 3l \\ v_{i+3l-q} & \text{if } q - 3l + 1 \leq i \leq q \\ \end{cases}
\]

For \( n = 1, m = 4 \) and \( q = 2 \times 4 + 1 = 9 = 3 \cdot 3^1 \). Then \( \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\} \) is the vertex set of \( K_9 \) have \( \frac{q(q - 1)}{2} = \frac{9 \times 8}{2} = 4 \cdot 9 = 36 \) edges. Since \( m = 4 \), we can partition the edges of \( K_9 \) into 4 edge disjoint sets, each set containing 9 edges by the above rule and their permutations are given as follows.

For \( j = 1 \), \( f_1 = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 \\ v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_1 \end{pmatrix} = (v_1v_2v_3v_4v_5v_6v_7v_8v_9) \)

For \( j = 2 \), \( f_2 = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 \\ v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_1 & v_2 \end{pmatrix} = (v_1v_3v_4v_5v_6v_7v_8v_9) \)

For \( j = 4 \), \( f_4 = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 \\ v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_1 & v_2 \end{pmatrix} = (v_1v_3v_5v_7v_9v_2v_4v_6) \)

Again, for \( l = 1 \), \( f_1 = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 \\ v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_1 & v_2 & v_3 \end{pmatrix} = (v_1v_4v_7)(v_2v_5v_8)(v_3v_6v_9) \)

\[32\]
From these permutations, we get 4 edge disjoint sets

\[ E_1 = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_7, v_7v_8, v_8v_9, v_9v_1\} \]

\[ E_2 = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_7, v_7v_8, v_8v_9, v_9v_1\} \]

\[ E_3 = \{v_1v_4, v_2v_5, v_3v_6, v_4v_7, v_5v_8, v_6v_9, v_7v_1, v_8v_2, v_9v_3\} \]

and

\[ E_4 = \{v_1v_5, v_2v_6, v_3v_7, v_4v_8, v_5v_9, v_6v_1, v_7v_2, v_8v_3, v_9v_4\} \]

Adjoining edges of \(E_1, E_2\) and \(E_4\), we get 3 edge disjoint circulant graphs \(C_9(1)\), \(C_9(2)\) and \(C_9(4)\) for the jumps \(j = 1, j = 2\) and \(j = 4\) respectively [Figure-3]. Also, adjoining the edges of \(E_3\), we get a circulant graph \(C_9(3)\) for the jump \(j = 3\) respectively [Figure-3]. Also, adjoining the edges of \(E_3\), we get a circulant graph \(C_9(3)\) for the jump \(j = 3\) respectively [Figure-3].

For \(n = 2, m = 13\) and \(q = 2 \times 13 + 1 = 27 = 3 \cdot 3^2\). Then \(\{v_1, v_2, \ldots, v_{27}\}\) is the vertex set of the complete graph \(K_{27}\) and have \(\frac{q(q-1)}{2} = \frac{27 \times 26}{2} = 13 \times 27 = 351\) edges. Since \(m = 13\), we can partition the edges of \(K_{27}\) into 13 edge disjoint sets, each set containing 27 edges, by the rule defined above.

For \(j = 1\), \(f_1 = (v_1v_2v_3v_4v_5v_6v_7v_8v_9v_{10}v_{11}v_{12}v_{13}v_{14}v_{15}v_{16}v_{17}v_{18}v_{19}v_{20}v_{21}v_{22}v_{23}v_{24}v_{25}v_{26}v_{27})\)

For \(j = 2\), \(f_2 = (v_1v_2v_3v_4v_5v_6v_7v_8v_9v_{10}v_{11}v_{12}v_{13}v_{14}v_{15}v_{16}v_{17}v_{18}v_{19}v_{20}v_{21}v_{22}v_{23}v_{24}v_{25}v_{26}v_{27})\)

For \(j = 3\), \(f_3 = (v_1v_2v_3v_4v_5v_6v_7v_8v_9v_{10}v_{11}v_{12}v_{13}v_{14}v_{15}v_{16}v_{17}v_{18}v_{19}v_{20}v_{21}v_{22}v_{23}v_{24}v_{25}v_{26}v_{27})\)

For \(j = 4\), \(f_4 = (v_1v_2v_3v_4v_5v_6v_7v_8v_9v_{10}v_{11}v_{12}v_{13}v_{14}v_{15}v_{16}v_{17}v_{18}v_{19}v_{20}v_{21}v_{22}v_{23}v_{24}v_{25}v_{26}v_{27})\)

Again,

For \(l = 1\), \(f_l = (v_1v_2v_3v_4v_5v_6v_7v_8v_9v_{10}v_{11}v_{12}v_{13}v_{14}v_{15}v_{16}v_{17}v_{18}v_{19}v_{20}v_{21}v_{22}v_{23}v_{24}v_{25}v_{26}v_{27})\)

For \(l = 2\), \(f_l = (v_1v_2v_3v_4v_5v_6v_7v_8v_9v_{10}v_{11}v_{12}v_{13}v_{14}v_{15}v_{16}v_{17}v_{18}v_{19}v_{20}v_{21}v_{22}v_{23}v_{24}v_{25}v_{26}v_{27})\)

For \(l = 3\), \(f_l = (v_1v_2v_3v_4v_5v_6v_7v_8v_9v_{10}v_{11}v_{12}v_{13}v_{14}v_{15}v_{16}v_{17}v_{18}v_{19}v_{20}v_{21}v_{22}v_{23}v_{24}v_{25}v_{26}v_{27})\)

For \(l = 4\), \(f_l = (v_1v_2v_3v_4v_5v_6v_7v_8v_9v_{10}v_{11}v_{12}v_{13}v_{14}v_{15}v_{16}v_{17}v_{18}v_{19}v_{20}v_{21}v_{22}v_{23}v_{24}v_{25}v_{26}v_{27})\)
From these permutations we can define corresponding edge disjoint sets \( E_1, E_2, E_3, E_4, E_5, E_6, E_7, E_8, E_9, E_{10}, E_{11}, E_{12} \) and \( E_{13} \) respectively.

Adjoining edges of \( E_1, E_2, E_4, E_5, E_7, E_8, E_{10}, E_{11}, \) and \( E_{13} \), we can construct \( 3^2 = 9 \) edge disjoint circulant graphs \( C_{27}(1), C_{27}(2), C_{27}(4), C_{27}(5), C_{27}(7), C_{27}(8), C_{27}(10), C_{27}(11) \) and \( C_{27}(13) \) for the jumps \( j = 1, j = 2, j = 4, j = 5, j = 7, j = 8, j = 10, j = 11 \) and \( j = 13 \) respectively. Also, adjoining edges of \( E_4, E_6, E_9 \) and \( E_{12} \), we get a circulant graph \( C_{27}(3,6,9,12) \) for the jump \( 3l \) where \( 1 \leq l \leq 4 \). Thus the complete graph \( K_{27} \) can be decomposed into \( 3^2 = 9 \) edge disjoint circulant graph \( C_{27}(j) \) where \( j \) is the jump such that \( 1 \leq j \leq 13, j \neq 3, 6, 9, 12 \) and a circulant graph \( C_{27}(3,6,9,12) \).

Let \( n = k \) so that \( m = \frac{1}{2}(3^{k+1} - 1) \) and \( q = 2m + 1 = 3 \cdot 3^k = 3^{k+1} \). Then \( \{v_1, v_2, v_3, \ldots, v_q\} \) is the vertex set the complete graph \( K_q = K_{3^{k+1}} \) and it has \( \frac{q(q-1)}{2} = \frac{3^{k+1}(3^{k+1} - 1)}{2} \) edges.

Since \( m = \frac{1}{2}(3^{k+1} - 1) \), so we can partition edges of \( K_q = K_{3^{k+1}} \) into \( \frac{1}{2}(3^{k+1} - 1) \) distinct sets, each set containing \( 3^{k+1} \) edges, given by

(i) when \( 1 \leq j \leq \frac{1}{2}(3^{k+1} - 1) \) but \( j \neq 3l \) for \( 1 \leq l \leq \frac{1}{2}(3^{k+1} - 1) \)

\[ f_j(v_i) = \begin{cases} v_{i+j} & \text{if } 1 \leq i \leq q - j \\ v_{i+j-q} & \text{if } q - j + 1 \leq i \leq q \end{cases} \]

and (ii) when \( 1 \leq l \leq \frac{1}{2}(3^{k+1} - 1) \)

\[ f_{3l}(v_i) = \begin{cases} v_{i+3l} & \text{if } 1 \leq i \leq q - 3l \\ v_{i+3l-q} & \text{if } q - 3l + 1 \leq i \leq q \end{cases} \]

Now, we have edge disjoint sets \( E_1, E_2, E_3, E_4, \ldots, E_{\frac{1}{2}(3^{k+1} - 1)} \) and their corresponding permutations are \( f_1, f_2, f_3, f_4, \ldots, f_{\frac{1}{2}(3^{k+1} - 1)} \).

Adjoining edges of \( E_1, E_2, E_4, E_5, E_7, \ldots, E_{\frac{1}{2}(3^{k+1} - 3)}, E_{\frac{1}{2}(3^{k+1} - 1)} \), we can construct \( 3^k \) edge disjoint circulant graphs \( C_{3^{k+1}}(j) \) where \( j \) is the jump and \( 1 \leq j \leq \frac{1}{2}(3^{k+1} - 1) \), \( j \neq 3l \) for \( 1 \leq l \leq \frac{1}{2}(3^{k+1} - 1) \). Also, adjoining edges of \( E_3, E_6, E_9, \ldots, E_{\frac{1}{2}(3^{k+1} - 7)}, E_{\frac{1}{2}(3^{k+1} - 3)} \), we get a circulant graph \( C_{3^{k+1}}(3,6,9,\ldots,3l) \) for the jump \( 1 \leq l \leq \frac{1}{2}(3^{k+1} - 1) \).

Now, adding \( 2 \cdot 3^{k+1} \) additional vertices to the vertex set and \( \left( 4 \cdot 9^{k+1} - 3^{k+1} \right) \) additional edges to the edge set of the complete graph \( K_{3^{k+1}} \). Then \( \{v_1, v_2, v_3, \ldots, v_{3^{k+1}}, v_{3^{k+1}+1}, \ldots, v_{3^{k+1}+2}, v_{3^{k+1}+3}\} \) will be the vertex set for the complete graph \( K_{3^{k+1}} = K_{3^{k+1}} \) i.e., \( K_q = K_{3^{k+1}} \) and \( q = 3^{k+2} \Rightarrow 2m + 1 = 3^{k+2} \Rightarrow m = \frac{1}{2}(3^{k+2} - 1) = \frac{1}{2} \left( 3^{k+1} + 1 \right) \).
The complete graph $K_{3^{t+2}}$ has $\frac{g(q-1)}{2} = \frac{3^{k+2}(3^{k+2} - 1)}{2} = \frac{3^{(k+1)+1}(3^{(k+1)+1} - 1)}{2}$ edges. These edges can be partitioned into $\frac{1}{2}(3^{k+2} - 1) = \frac{1}{2}\{3^{(k+1)+1} - 1\}$ disjoint sets and their respective permutations are given by

(i) when $1 \leq j \leq \frac{1}{2}(3^{k+2} - 1)$ but $j \neq 3l$ for $1 \leq l \leq \frac{1}{2}(3^{k+1} - 1)$

$$f_j(v_i) = \begin{cases} v_{i+j} & \text{if } 1 \leq i \leq q - j \\ v_{i+j-q} & \text{if } q - j + 1 \leq i \leq q \end{cases}$$

and (ii) when $1 \leq l \leq \frac{1}{2}(3^{k+1} - 1)$, $f_{3l}(v_i) = \begin{cases} v_{i+3l} & \text{if } 1 \leq i \leq q - 3l \\ v_{i+3l-q} & \text{if } q - 3l + 1 \leq i \leq q \end{cases}$

Now, we can have $\frac{1}{2}\{3^{(k+1)+1} - 1\}$ edge disjoint sets $E_1$, $E_2$, $E_3$, $E_4$, ..., $E_{\frac{1}{2}(3^{k+2} - 1)}$ and their respective permutations are $f_1$, $f_2$, $f_3$, $f_4$, ..., $f_{\frac{1}{2}(3^{k+2} - 1)}$. Adjoining edges of $E_1$, $E_2$, $E_4$, $E_5$, $E_7$, $E_8$, $E_{12}$, $E_{13}$, $E_{15}$, $E_{16}$, $E_{17}$, $E_{18}$, $E_{21}$, $E_{22}$, we can construct $3^{(k+1)}$ edge disjoint circulant graph $C_{3^{t+2}}(j)$ i.e., $C_{3^{(k+1)+1}}(j)$ where $j$ is the jump and $1 \leq j \leq \frac{1}{2}(3^{(k+1)+1} - 1)$, $j \neq 3l$ for $1 \leq l \leq \frac{1}{2}(3^{k+1} - 1)$. Also, adjoining edges of $E_3$, $E_6$, $E_9$, $E_{11}$, $E_{14}$, $E_{19}$, $E_{20}$, $E_{24}$, $E_{27}$, $E_{30}$, we have a circulant graph $C_{3^{(k+1)+1}}(3, 6, 9, ..., 3l)$ for the jump $1 \leq l \leq \frac{1}{2}(3^{k+1} - 1)$. Thus the result is true for $n = k + 1$ whenever it is true for $n = k$. Hence by principle of mathematical induction, the result is true for all $n \in N$. In this discussion, we observed that each edge disjoint circulant graph $C_q(j)$ for the jump $1 \leq j \leq \frac{1}{2}(3^{n+1} - 1)$, $j \neq 3l$ for $1 \leq l \leq \frac{1}{2}(3^n - 1)$ are regular of degree two. In other words they are cycle $C_q$ of length $q$ and therefore they represent $3^n$ edge disjoint Hamiltonian circuits $n \geq 1$. Further the circulant graph $C_q(3, 6, 9, ..., 3l)$ for the jump $1 \leq l \leq \frac{1}{2}(3^n - 1)$, as we have observed, can be decomposed into $\frac{1}{2}(3^n - 1)$ edge disjoint circulant graphs $C_q(3t)$ for the jump $3t$, $1 \leq t \leq \frac{1}{2}(3^n - 1)$ for all $n \geq 1$. If the jump is

- $3t = 3^n$, then $3^n$ $K_3$-decomposition
- $3t = 3^{n-1}$, then $3^n$ $C_9$-decomposition
- $3t = 3^{n-2}$, then $3^n$ $C_{27}$-decomposition
- $3t = 3^{n-3}$, then $3^n$ $C_{81}$-decomposition
- $3t = 3^{n-4}$, then $3^n$ $C_{243}$-decomposition
- $3t = 3^n$, then $3^n$ $C_{3^{n-1}}$-decomposition
- $3t = 3^{n-2}$, then $3^n$ $C_{3^{n-2}}$-decomposition
- $3t = 3^{n-1}$, then $3^n$ $C_{3^{n-3}}$-decomposition
- $3t = 3^n$, then $3^n$ $C_{3^{n-4}}$-decomposition
Clearly above cycles does not represent Hamiltonian circuits. So in this discussion, number of edge disjoint Hamiltonian circuits in the complete graph \( K_q \) is \( n \) if \( q = 3 \cdot 3^n \) for \( n \geq 1 \).

(e) Let \( K_{2m+1} \) be a complete graph for all \( m \geq 2 \). Suppose that \( r = 2m+1 = 3s \) where \( s \geq 5 \) is a prime number and \( m = \frac{3s-1}{2} \). Then \( V = \{v_1, v_2, v_3, \ldots, v_r\} \) is the vertex set of the complete graph \( K_r \), i.e., \( K_{3s} \) and have \( \frac{r(r-1)}{2} \) edges.

Let us define \( f_j : V(K_r) \rightarrow V(K_r) \) as follows:

(i) when \( 1 \leq j \leq \frac{1}{2}(3s-1) \) but \( j \neq s \) and \( j \neq 3e \) for \( 1 \leq e \leq \frac{1}{2}(s-1) \)

\[
f_j(v_i) = \begin{cases} 
  v_{i+j} & \text{if } 1 \leq i \leq r-j \\
  v_{i+j-r} & \text{if } r-j+1 \leq i \leq r
\end{cases}
\]

(ii) when \( 1 \leq e \leq \frac{1}{2}(s-1) \), \( f_{3e}(v_i) = \begin{cases} 
  v_{i+3e} & \text{if } 1 \leq i \leq r-3e \\
  v_{i+3e-r} & \text{if } r-3e+1 \leq i \leq r
\end{cases}
\]

and (iii) \( f_s(v_i) = \begin{cases} 
  v_{i+s} & \text{if } 1 \leq i \leq r-s \\
  v_{i+s-r} & \text{if } r-s+1 \leq i \leq r
\end{cases}
\]

For \( s = 5 \), a prime number, we have \( r = 3 \times 5 = 15 \) and \( m = \frac{3 \times 5 - 1}{2} = 7 \).

Then \( \{v_1, v_2, v_3, \ldots, v_{14}, v_{15}\} \) is the vertex set of the complete graph \( K_{15} \) and have \( \frac{r(r-1)}{2} = 15 \times 14 = 7 \times 15 = 105 \) edges. Since \( m = 7 \), we can partition the edges of \( K_{15} \) into 7 disjoint sets, each set containing 15 edges and their corresponding permutations by the above rule are as follows:

For \( j = 1 \), \( f_1 = (v_1v_2v_3v_4v_5v_6v_7v_8v_9v_{10}v_{11}v_{12}v_{13}v_{14}v_{15}) \) and

\( E_1 = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_7, v_7v_8, v_8v_9, v_9v_{10}, v_{10}v_{11}, v_{11}v_{12}, v_{12}v_{13}, v_{13}v_{14}, v_{14}v_{15}, v_{15}v_1\} \)

For \( j = 2 \), \( f_2 = (v_1v_3v_5v_7v_9v_{11}v_{13}v_{15}v_2v_4v_6v_8v_{10}v_{12}v_{14}) \) and

\( E_2 = \{v_1v_3, v_3v_5, v_5v_7, v_7v_9, v_9v_{11}, v_{11}v_{13}, v_{13}v_{15}, v_{15}v_1, v_1v_2, v_2v_4, v_4v_6, v_6v_8, v_8v_{10}, v_{10}v_{12}, v_{12}v_{14}, v_{14}v_2\} \)

For \( j = 3 \), \( f_3 = (v_1v_3v_5v_7v_9v_{11}v_{13}v_{15}v_2v_4v_6v_8v_{10}v_{12}v_{14}v_3v_5v_7v_9v_{11}v_{13}v_{15}v_1) \) and

\( E_3 = \{v_1v_3, v_3v_5, v_5v_7, v_7v_9, v_9v_{11}, v_{11}v_{13}, v_{13}v_{15}, v_{15}v_1, v_1v_2, v_2v_4, v_4v_6, v_6v_8, v_8v_{10}, v_{10}v_{12}, v_{12}v_{14}, v_{14}v_3, v_3v_5, v_5v_7, v_7v_9v_{11}v_{13}v_{15}v_1\} \)

For \( e = 1 \), \( f_5 = (v_1v_3v_5v_7v_{11}v_{13}v_{15}v_2v_4v_6v_8v_{10}v_{12}v_{14}) \) and

\( E_5 = \{v_1v_3, v_3v_5, v_5v_7, v_7v_9, v_9v_{11}, v_{11}v_{13}, v_{13}v_{15}, v_{15}v_1, v_1v_2, v_2v_4, v_4v_6, v_6v_8, v_8v_{10}, v_{10}v_{12}, v_{12}v_{14}, v_{14}v_3, v_3v_5, v_5v_7, v_7v_9, v_9v_{11}v_{13}v_{15}v_1\} \)

For \( e = 2 \), \( f_6 = (v_1v_3v_5v_7v_{11}v_{13}v_{15}v_2v_4v_6v_8v_{10}v_{12}v_{14}v_3v_5v_7v_9v_{11}v_{13}v_{15}v_1) \) and

\( E_6 = \{v_1v_3, v_3v_5, v_5v_7, v_7v_9, v_9v_{11}, v_{11}v_{13}, v_{13}v_{15}, v_{15}v_1, v_1v_2, v_2v_4, v_4v_6, v_6v_8, v_8v_{10}, v_{10}v_{12}, v_{12}v_{14}, v_{14}v_3, v_3v_5, v_5v_7, v_7v_9, v_9v_{11}v_{13}v_{15}v_1\} \)

For \( s = 5 \), \( f_5 = (v_1v_3v_5v_7v_{11}v_{13}v_{15}v_2v_4v_6v_8v_{10}v_{12}v_{14}v_3v_5v_7v_9v_{11}v_{13}v_{15}v_1) \) and

\( E_5 = \{v_1v_3, v_3v_5, v_5v_7, v_7v_9, v_9v_{11}, v_{11}v_{13}, v_{13}v_{15}, v_{15}v_1, v_1v_2, v_2v_4, v_4v_6, v_6v_8, v_8v_{10}, v_{10}v_{12}, v_{12}v_{14}, v_{14}v_3, v_3v_5, v_5v_7, v_7v_9, v_9v_{11}v_{13}v_{15}v_1\} \)

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Adjoining edges of $E_1$, $E_2$, $E_4$ and $E_7$, we can construct $s-1=5-1=4$ edge disjoint circulant graphs $C_{15}(1)$, $C_{15}(2)$, $C_{15}(4)$ and $C_{15}(7)$ respectively for jumps $j=1$, $j=2$, $j=4$ and $j=7$ [Figure-5]. Also, adjoining edges of $E_3$ and $E_6$, we can construct a circulant graph $C_{15}(3,6)$ [Figure-6]. Again, adjoining the edges of $E_5$, we can construct a circulant graph $C_{15}(5)$ for the jump $s=5$ [Figure-7].

Thus the complete graph $K_{15}$ can be decomposed into $s-1=5-1=4$ edge disjoint circulant graphs $C_{15}(j)$ for $1\leq j \leq 7$, $j \neq 3,5,6$ and a circulant graph $C_{15}(3,6)$ and a circulant graph $C_{15}(5)$ for $s=5$. Similarly, we can show for $s=7,11,13,\ldots$ etc.

Let $s=k$, a prime number. Then $r=3k$, $m=\frac{3k-1}{2}$ and the complete graph $K_{3k}$ has $\frac{r(r-1)}{2} = \frac{3k(3k-1)}{2}$ edges. Since $m=\frac{3k-1}{2}$, so we can partition the edges of $K_{3k}$ into $\frac{1}{2}(3k-1)$ edge disjoint sets, each set containing $3k$ edges and their respective permutations are given by

(i) when $1 \leq j \leq \frac{1}{2}(3k-1)$ but $j \neq k$ and $j \neq 3e$ for $1 \leq e \leq \frac{1}{2}(k-1)$

$$f_{j}(v_{i}) = \begin{cases} 
 v_{i+j}, & \text{if } 1 \leq i \leq r-j \\
 v_{i+j-r}, & \text{if } r-j+1 \leq i \leq r
\end{cases}$$

(ii) when $1 \leq e \leq \frac{1}{2}(k-1)$, $f_{3e}(v_{i}) = \begin{cases} 
 v_{i+3e}, & \text{if } 1 \leq i \leq r-3e \\
 v_{i+3e-r}, & \text{if } r-3e+1 \leq i \leq r
\end{cases}$

and (iii) $f_{k}(v_{i}) = \begin{cases} 
 v_{i+k}, & \text{if } 1 \leq i \leq r-k \\
 v_{i+k-r}, & \text{if } r-k+1 \leq i \leq r
\end{cases}$
Now, we can have edge disjoint sets \(E_1, E_2, E_3, E_4, \ldots, E_{\frac{1}{2}(3k-1)}\) and their respective permutations are \(f_1, f_2, f_3, f_4, \ldots, f_{\frac{1}{2}(3k-1)}\). Adjoining edges of \(E_j\)'s where \(1 \leq j \leq \frac{1}{2}(3k-1)\); \(j \neq k\) and \(j \neq k\) for \(1 \leq e \leq \frac{1}{2}(k-1)\), we can construct \(s-1=k-1\) edge disjoint circulant graphs \(C_{3k}(j)\) where \(j\) is the jump. Again, adjoining the edges of \(E_{3e}\)'s, \(1 \leq e \leq \frac{1}{2}(k-1)\) we can construct a circulant graph \(C_{3k}(3,6,9,\ldots,3e)\) for the jump \(3e\). Also, adjoining the edges of \(E_k\), we can construct a circulant graph \(C_{3k}(k)\) for the jump \(k\).

In this discussion we observed that each edge disjoint circulant graphs \(C_{r}(j)\) for the jump \(1 \leq j \leq \frac{1}{2}(3s-1), j \neq s\) and \(j \neq 3e\) for \(1 \leq e \leq \frac{1}{2}(s-1)\) where \(s \geq 5\) is a prime, are regular of degree two. In other words they are cycles \(C_{r}\) of length \(r\) and therefore they represent edge disjoint Hamiltonian circuits. Again the circulant graph \(C_{r}(3,6,9,\ldots,3e)\) for the jump \(3e\) where \(1 \leq e \leq \frac{1}{2}(s-1)\) can be decomposed into \(m-s\) edge disjoint circulant graphs \(3t\) where \(1 \leq t \leq \frac{1}{2}(s-1)\) and these circulants are regular of degree two each and they can be further decomposed into \((m-1)\) cycles \(C_{s}\) of length \(s<r\) and so they do not represent Hamiltonian circuits. Also, the circulant graph \(C_{r}(s)\) for the jump \(s\) is regular of degree two and from which we can have \(s\) \(K_{3}\)-decompositions and they do not represent a Hamiltonian circuit. So in this discussion, number of edge disjoint Hamiltonian circuits in the complete graph \(K_{r}\) is \(s-1\) if \(r=3s\), \(s \geq 5\) is a prime.

4. ALGORITHM

**Input:** Let \(K_{2m+1}\) for \(m \geq 2\) be a complete graph.

**Output:** Find least cost route.

**Case 1:** When \(p = 2m+1\) is a prime number and \((2m+1)\) consecutive greatest weights are incident at different vertices of the complete graph \(K_{2m+1}\) for \(m \geq 2\) along edges of a cycle \(C_{2m+1}\), next \((2m+1)\) greatest weights incident at different vertices along edges of a cycle \(C_{2m+1}\) other than the previous one, next \((2m+1)\) greatest weights incident at different vertices along edges of a cycle \(C_{2m+1}\) other than the previous two cycles and so on.

**Step 1:** Study the graph \(K_{2m+1}\) for \(m \geq 2\).

**Step 2:** Delete \((2m+1)\) number of consecutive greatest weights incident at different vertices along the edges of a cycle \(C_{2m+1}\) of the complete graph \(K_{2m+1}\) for \(m \geq 2\).

**Step 3:** Observed that the graph obtained in Step 2 is a regular sub graph of the complete graph \(K_{2m+1}\) of degree 2. Otherwise go to Step 6.

**Step 4:** Observed that the graph obtained in Step 3 is a cycle \(C_{2m+1}\) for \(m \geq 2\) attached with minimum weights and this is the required Hamiltonian circuit i.e., least cost route.

**Step 5:** Stop

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Step 6: From the graph obtained in Step 2, delete another \((2m+1)\) number of edges attached with consecutive greatest weights other than \((2m+1)\) consecutive greatest weights already deleted in Step 2.

Step 7: Go to Step 3 to Step 5. Otherwise go to Step 8.

Step 8: Repeat Step 6 till reaching the stage of Step 3 to Step 4 and go to Step 5.

Case 2: When \(q = 2m + 1 = 3 \cdot 3^n\) for \(n \geq 1\) and \(3^{n+1}\) number of consecutive greatest weights incident at different vertices along edges of \(3^n\) edge disjoint \(K_3\), next \(3^{n+2}\) number of consecutive greatest weights attached with edges of \(3^n\) edge disjoint cycles \(C_{3^2}\), next \(3^{n+3}\) number of consecutive greatest weights attached with edges of \(3^n\) edge disjoint cycles \(C_{3^3}\) and so on the remaining \(3^{2n+1}\) number of consecutive greatest weights are attached with edges of \(3^n\) circuits \(C_{3^{n+1}}\).

Step 9: Delete \(3^{n+1}\) number of consecutive greatest weights incident at different vertices along edges of \(3^n\) edge disjoint \(K_3\).

Step 10: Observed that the graph obtained in Step 9 is a regular sub graph of the complete graph \(K_{3^{3^n}}\) of degree \(2 \cdot 3^n\) for \(n \geq 1\).

Step 11: From the graph obtained in Step 11, delete another \(3^{n+1}\) number of edges forming the circuit \(C_{3^{n+1}}\) attached with consecutive greatest weights other than \(3^{n+1}\) consecutive greatest weights already deleted in Step 9, successively \((3^n - 1)\) times.

Step 12: Observed that the graph obtained in Step 11 is a regular sub graph of the complete graph \(K_{3^{3^n}}\) of degree 2 for \(n \geq 1\). Otherwise go to Step 15.

Step 13: Observed that graph obtained in Step 12 is a circuit \(C_{3^{n+1}}\) attached with minimum weights, which is the required least cost route.

Step 14: Go to Step 5.

Step 15: Observed that the graph obtained in Step 10 is a regular sub graph of the complete graph \(K_{3^{3^n}}\) of degree \(3^{(3^n - 1)}\) for \(n \geq 1\).

Step 16: From the graph obtained in Step 15, delete another \(3^{n+2}\) number of consecutive greatest weights other than the greatest weights already deleted in Step 11, incident at different vertices forming \(3^n\) edge disjoint cycles \(C_{3^2}\) for \(n \geq 1\).

Step 17: Go to Step 10 to Step 14. Otherwise go to Step 18.

Step 18: From the graph obtained in Step 17, delete another \(3^{n+3}\) number of consecutive greatest weights other than the greatest weights already deleted in Step 16, incident at different vertices forming \(3^n\) edge disjoint cycles \(C_{3^3}\) for \(n \geq 1\).

Step 19: Go to Step 10 to Step 14. Otherwise go to Step 20.

Step 20: From the graph obtained in Step 18, delete another \(3^{n+4}\) number of consecutive greatest weights other than the greatest weights already deleted in Step 18, incident at different vertices forming \(3^n\) edge disjoint cycles \(C_{3^4}\) for \(n \geq 1\) and continue this process till the graph becomes regular of degree \(2 \cdot 3^n\) for \(n \geq 1\).

Step 21: Go to Step 10 to Step 14.
**Case 3:** When \( r = 2m + 1 = 3s, \ s \geq 5 \) is a prime and \( 3s \) number of consecutive greatest weights incident at different vertices forming \( s \) edge disjoint \( K_3 \), next \( 3s(m-s) \) number of consecutive greatest weights incident at different vertices forming \( 3(m-s) \) edge disjoint cycles \( C_s \) and the remaining \( 3s(s-1) \) number of consecutive greatest weights incident at different vertices forming \( (m-s) \) edge disjoint cycles \( C_{3s} \).

**Step 22:** Delete \( 3s \) number of consecutive greatest weights incident at different vertices forming \( s \) edge disjoint \( K_3 \) for \( s \geq 5 \).

**Step 23:** Observed that the graph obtained in Step 22 is a regular sub graph of the complete graph \( K_{3s} \) of degree \( 3(s-1) \) for \( s \geq 5 \).

**Step 24:** From the graph obtained in Step 23, delete another \( 3s(m-s) \) number of edges forming \( 3(m-s) \) edge disjoint circuits \( C_s \) attached with greatest consecutive weights, other than \( 3s \) number of consecutive greatest weights already deleted in Step 22.

**Step 25:** Observed that the graph obtained in Step 24 is a regular sub graph of the complete graph \( K_{3s} \) of degree \( 2(s-1) \) having \( 3s(s-1) \) number of remaining edges for \( s \geq 5 \).

**Step 26:** From the graph obtained in Step 25, delete another \( 3s \) number of edges forming the cycle \( C_{3s} \) attached with consecutive greatest weights which are different from greatest weights already deleted in Step 24.

**Step 27:** Observed that the graph obtained in Step 26 is a regular sub graph of the complete graph \( K_{3s} \) of degree \( 2(s-2) \) for \( s \geq 5 \).

**Step 28:** Repeat Step 26 on the graph obtained in Step 23 for \( (s-3) \) times where \( s \geq 5 \).

**Step 29:** Observed that the graph obtained in Step 28 is a cycle \( C_{3s} \) attached with minimum weights and this is the required Hamiltonian circuit i.e., least cost route.

**Step 30:** Go to Step 5

### 5. EXPERIMENTAL RESULTS FOR ALGORITHM

**Case 1:** The following cost matrix (Table-1) is considered for seven cities (i.e., for \( m = 3 \) of \( K_{2m+1} \) and \( 2m + 1 = 7 \), a prime)

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
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<tbody>
<tr>
<td>A</td>
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<td>B</td>
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<td>( \infty )</td>
<td>23</td>
<td>13</td>
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<tr>
<td>C</td>
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<td>( \infty )</td>
<td>27</td>
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<td>40</td>
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<tr>
<td>D</td>
<td>35</td>
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<td>27</td>
<td>( \infty )</td>
<td>30</td>
<td>16</td>
<td>37</td>
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<tr>
<td>E</td>
<td>51</td>
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<td>6</td>
<td>30</td>
<td>( \infty )</td>
<td>31</td>
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<td>F</td>
<td>18</td>
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<td>16</td>
<td>31</td>
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<td>33</td>
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<td>G</td>
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<td>40</td>
<td>37</td>
<td>7</td>
<td>33</td>
<td>( \infty )</td>
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</tbody>
</table>

**Table-1**
From the Table-1, we have a complete graph of seven vertices, which is shown in Figure-8.
Now, applying the Step 2 of the algorithm, we have a graph as shown in Figure-9.
Now, applying the Step 6 to Step 8 of the algorithm, we have a graph as shown in Figure-10 and this is the least cost route as \( A \rightarrow C \rightarrow E \rightarrow G \rightarrow B \rightarrow D \rightarrow F \rightarrow A \) with weights equal to 63.

Case 2: The following cost matrix (Table-2) is considered for nine cities (i.e., for \( m = 4 \) of \( K_{2m+1} \) and \( 2m+1 = 9 = 3 \cdot 3^1 \))

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<th>D</th>
<th>E</th>
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Table-2

From the Table-2, we have a complete graph of nine vertices, which is shown in Figure-11.
Now, applying the Step 9 of the algorithm, we delete 9 greatest weights attached with edges, \( AD = 81 \), \( DG = 84 \), \( AG = 86 \), \( BE = 88 \), \( EH = 90 \), \( BH = 91 \), \( CF = 93 \), \( FI = 95 \), \( CI = 97 \) forming 3 edge disjoint \( K_3 \). Then we have a graph as shown in Figure-12.

Now, applying the Step10 to Step14 of the algorithm, we delete a cycle \( C_9 \) whose edges are attached with greatest weights \( AE = 59 \), \( EI = 61 \), \( ID = 64 \), \( DH = 69 \), \( HC = 70 \), \( CG = 73 \), \( GB = 76 \), \( BF = 78 \), \( FA = 80 \), and then we delete another cycle \( C_9 \) whose edges are attached with next greatest weights \( AB = 31 \), \( BC = 34 \), \( CD = 39 \), \( DE = 41 \), \( EF = 47 \), \( FG = 49 \), \( GH = 52 \), \( HI = 55 \), \( IA = 57 \). Then we have a graph as shown in Figure-13 and this is the least cost route of the traveler as \( A \rightarrow C \rightarrow E \rightarrow G \rightarrow I \rightarrow B \rightarrow D \rightarrow F \rightarrow H \rightarrow A \) with weights equal to 160.
Case 3: The following cost matrix (Table-3) is considered for fifteen cities (i.e., for $m = 7$ of $K_{2m+1}$ and $2m + 1 = 15 = 3 \times 5$)

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</table>

Table: 3

From the Table-3, we have a complete graph of nine vertices, which is shown in Figure-14.
Now, applying the Step 22 of the algorithm, we delete 15 edges forming 5 edge disjoint $K_1$ attached with greatest weights $AE=166$, $FK=167$, $AK=169$, $BG=170$, $GL=173$, $BL=175$, $CH=176$, $HM=178$, $CM=180$, $DI=181$, $IN=183$, $DN=184$, $EJ=185$, $JO=187$, $EO=190$. Then we have a regular sub graph of degree 12 as shown in Figure-15.
Now, applying the Step 24 of the algorithm, we delete 30 edges forming 6 edge disjoint cycles $C_6$ attached with next greatest weights $AG=143, GM=144, MD=146, DJ=147, AJ=149, BH=150, HN=146, DJ=147, AJ=149, BH=150, HN=152, NE=153, EK=155, BK=156, CI=158, IO=159, OF=161, FL=162, CL=163, AD=122, DG=123, GJ=125, JM=127, MA=128, BE=129, EH=130, HK=132, KN=133, NB=135, CF=136, FI=138, IL=139, LO=140, OC=141. Then we have a regular sub graph of $K_{15}$ of degree 8 as shown in Figure-16.

![Figure-16](image1)

Now, applying the Step 26 of the algorithm, we delete 15 edges forming the cycle $C_{15}$ attached with next greatest weights $AB=94, BC=96, CD=97, DE=99, EF=101, FG=104, GH=105, HI=109, IJ=110, JK=112, KL=115, LM=116, MN=118, NO=119, OA=121$. Then we have a regular sub graph of $K_{15}$ of degree 6 as shown in Figure-17.

![Figure-17](image2)
Applying the Step 28 of the algorithm, we delete 30 edges forming 2 edge disjoint cycles $C_{15}$ attached with next greatest weights $AC=69$, $CE=70$, $EG=71$, $GI=73$, $IK=76$, $KM=77$, $MO=79$, $OB=80$, $BD=81$, $DF=83$, $FH=86$, $HJ=87$, $LN=90$, $NA=93$, $AE=44$, $EI=45$, $IM=46$, $MB=48$, $BF=50$, $FJ=51$, $JN=53$, $NC=54$, $CG=55$, $GK=57$, $KO=59$, $OD=60$, $DH=61$, $HL=63$, $LA=66$. Then we have a graph as shown in Figure-18 and this is the least cost route as $A \rightarrow H \rightarrow O \rightarrow G \rightarrow N \rightarrow F \rightarrow M \rightarrow E \rightarrow L \rightarrow D \rightarrow K \rightarrow C \rightarrow J \rightarrow B \rightarrow I \rightarrow A$ with weights equal to 349.

6. CONCLUSION

Here we have discussed decomposition of complete graphs $K_{2m+1}$ for $m \geq 2$ into circulant graphs if $2m+1 = a$ prime number, $2m+1 = 3 \cdot 3^n, n \geq 1$ and $2m+1 = 3s, s \geq 5$ is a prime. However, one can study the other situation like $2m+1 = 5 \cdot 5^n, n \geq 1$ or $2m+1 = 5s, s \geq 7$ and find the circulant graph which may lead to study the Hamiltonian circuit related with the traveling salesman problems in near future.

REFERENCES


[4] Dalibor Froncek, Cyclic decomposition of complete graphs into $K_{m,n} + e$ : the missing case, Congressus Numerantium, 198 (2009), 111-118.


