SELF-GRAVITATING ELECTRODYNAMIC STABILITY OF ACCELERATING STREAMING FLUID CYLINDERS IN SELF-GRAVITATING TENUOUS MEDIUM

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ABSTRACT

The stability of electrodynamic self-gravitational of accelerating streaming fluid cylinder surrounded by tenuous medium has been studied for all axisymmetric and non-axisymmetric perturbation modes. The dispersion relation is derived, discussed and some reported works are recovered as limiting cases from it. The electro-dynamic forces interior and exterior the fluid cylinder are destabilizing. The self-gravitating forces are destabilizing for small axisymmetric domain while they are stabilizing in all other domains. The accelerating (periodic) streaming has a stabilizing tendency. The self-gravitating electrodynamic stable domains are enlarged due to the resonance of the periodic streaming and could overcome the instability character of the model, if it is so strong.

1. INTRODUCTION

The electrodynamic stability of cylindrical fluids has been the attention of several scientists (see Melcher et. al. (1971), and Nayyar et. al. (1970)) since the pioneering works of Kelly (1965). The effect of the electrodynamic force on the capillary force and other parameters has been examined by Radwan (1992), and Mestel (1994) & (1996).

In the last decades of 20\textsuperscript{th} century interests have been devoted to the self-gravitating instability of cylindrical models for their crucial applications. Chandrasekhar and Fermi (1953) were the first to discuss the stability of the full fluid cylinder under the influence of the self-gravitating force and did write about its importance with the oscillation of the spiral arm of galaxy, for more studies see also Chandrasekhar (1981), Radwan (2007) and Radwan & Hussain(2009). Hasan (2011) has discussed the stability of oscillating streaming fluid cylinder subject to combined effect of the capillary, self gravitating and electrodynamic forces for all axisymmetric and non axisymmetric perturbation modes. He (2012) studied the magnetodynamic stability of a fluid jet pervaded by
transverse varying magnetic field while its surrounding tenuous medium is penetrated by uniform magnetic field.

In the present work we investigate the electrodynamic self-gravitational stability of oscillating streaming fluid cylinder surrounded by self-gravitating tenuous medium of very slow motion.

2. FORMULATION OF THE PROBLEM

We consider a dielectric fluid cylinder of uniform cross-section of (radius \( R_0 \)) permittivity \( \varepsilon^{(1)} \) surrounded by tenuous medium of negligible motion of permittivity \( \varepsilon^{(2)} \). The fluid model is assumed to be incompressible, self-gravitating, inviscid and pervaded by uniform electric field

\[
E^{(1),(2)}_0 = (0,0,E_0)
\]

where \( E_0 \) is the intensity of the electric field in the fluid. The fluid streams with periodic streaming at time \( t \)

\[
u_0 = (0,0,U \cos \Omega t)
\]

where \( U \) is streaming velocity at time \( t = 0 \) while \( \Omega \) is the oscillation frequency of the velocity. The components of \( E^{(1),(2)}_0 \) and \( \nu_0 \) are considered along the cylindrical coordinates \((r, \phi, z)\) with the \( z \)-axis coinciding with the axis of the fluid cylinder. The cylindrical fluid model is acted by the combined effect of the pressure gradient, electrodynamic and self-gravitating forces.

The required basic equations are the combination of the electro-dynamic Maxwell equations, ordinary hydro-dynamic equations and Newtonian equation concerning the self-gravitating matter.

For the model under consideration, these equations are given as follows:

\[
\rho \left( \frac{\partial \nu}{\partial t} + (\nu \nabla) \nu \right) = -\nabla p + \rho \nabla V^{(1)} + \frac{1}{2} \nabla \varepsilon^{(1)} \left( E^{(1)} \cdot E^{(1)} \right)
\]

\[
\nabla \nu = 0 \quad , \quad \nabla \times E^{(1),(2)} = 0
\]

\[
\nabla \times (\varepsilon E)^{(1),(2)} = 0
\]

\[
\nabla^2 V^{(1)} = -4\pi G \rho^{(1)}
\]

\[
\nabla^2 V^{(2)} = 0
\]

This system of equations is solved in the unperturbed state with \( \nu_0 = (0,0,U) \), \( E^{(1),(2)}_0 = (0,0,E_0) \), and after applying the required boundary conditions at \( r = R_0 \), we get

\[
V^{(1)}_0 = -\pi G \rho r^2
\]
\[ V_0^{(2)} = 2\pi G \rho R_0^2 \ln \left( \frac{R_0}{r} \right) - \pi G \rho R_0^2 \] (10)

Sketch of the Periodic Streaming Fluid Model
3. PERTURBATION STATE

For small departures from the unperturbed state, every physical quantity $Q(r, \varphi, z; t)$ may be expressed as

$$Q(r, \varphi, z; t) = Q_0(r) + \eta(t)Q_1(r, \varphi, z) + \cdots$$  \hspace{1cm} (11)

where $Q$ stands for $u, p, V^{(1)}, V^{(2)}, E^{(1)}, E^{(2)}$ with $Q_0(r)$ is the value of $Q$ in the unperturbed state with $Q_1$ is small increment of $Q$ due to perturbation. Based on the expansion (11), the deformation in the cylindrical interface is given by

$$r = R_0 + \eta$$  \hspace{1cm} (12)

with

$$\eta = \eta(t) \exp \left( i (kz + m \varphi) \right)$$  \hspace{1cm} (13)

is the elevation of the surface wave measured from the unperturbed level.

By using the expansion (11) in the fundamental equations (3)-(8), the relevant perturbation equations are given as follows

$$\rho \left( \frac{\partial u}{\partial t} + (u_0 \cdot \nabla)u_1 \right) = -\nabla p_1 + \rho \dot{V}^{(1)}_1 + \varepsilon^{(1)} \nabla \left( 2\varepsilon_0 \cdot E^{(1)}_1 \right)$$  \hspace{1cm} (14)

$$\nabla \cdot u_1 = 0, \quad \nabla \cdot E^{(1),(2)}_1 = 0$$  \hspace{1cm} (15), (16)

$$\nabla \wedge \left( \varepsilon E^{(1),(2)}_1 \right) = 0$$  \hspace{1cm} (17)

$$\nabla^2 V^{(1)}_1 = 0$$  \hspace{1cm} (18)

$$\nabla^2 V^{(2)}_1 = 0$$  \hspace{1cm} (19)

Since the fluid is irrotational, we have

$$u_1 = -\nabla \phi_1$$  \hspace{1cm} (20)

By inserting this in (16), the velocity potential $\phi_1$ satisfies

$$\nabla^2 \phi_1 = 0$$  \hspace{1cm} (21)

and based on the expansion (11)-(13), the solution of (21) give
\[ \phi_1 = A^{(1)}_B(t) I_m(kr) \exp \left[ i (kz + m\phi) \right] \]  

(22)

with

\[ u_{tr} = \frac{\partial \eta}{\partial t} + ikU \eta \cos \Omega t \]  

(23)

From equation (14), we get

\[ \rho \left( \frac{\partial u_{tr}}{\partial t} + U \cos \Omega t \frac{\partial u_{tr}}{\partial z} \right) = -\frac{\partial \Pi_1}{\partial r} \]  

(24)

\[ \pi_1 = p_1 - \rho V^{(1)} - \varepsilon^{(1)} \left( E_0^{(1)} E_1^{(1)} \right) \]  

(25)

So

\[ \rho \frac{\partial}{\partial t} \left( \frac{\partial \eta}{\partial t} + ikU \eta \cos \Omega t \right) + ikU \rho \left( \frac{\partial \eta}{\partial t} + ikU \eta \cos \Omega t \right) = -\frac{\partial \Pi_1}{\partial r} \]

Taking into account

\[ \nabla^2 \Pi_1 = 0, \]

so, we have

\[ \Pi_1 = C_1(t) \eta(t) I_m(kr) \exp \left[ i (kz + m\phi) \right] \]  

(26)

Hence

\[ \rho \frac{\partial^2 \eta}{\partial t^2} + 2i \rho kU \cos \Omega t \frac{\partial \eta}{\partial t} - i \rho kU \eta \sin \Omega t - \rho k^2 U^2 \cos^2 \Omega t \]

(27)

\[ = -C_1 \eta_k l_m(x) \]

and

\[ \Pi_1 = -\frac{\rho}{k} \frac{\eta_k(x)}{I'_m(x)} \left( \frac{\partial^2 \eta}{\partial t^2} + 2i kU \cos \Omega t \frac{\partial \eta}{\partial t} \right) - i kU \eta \sin \Omega t - k^2 U^2 \cos^2 \Omega t \]  

(28)

Now, upon using equation (17) concerning \( E^{(1),(2)}_1 \), we have \( E^{(1),(2)}_1 = \nabla \psi^{(1),(2)}_1 \) so,

\[ \nabla^2 \psi^{(1),(2)}_1 = 0, \]

therefore

\[ \psi^{(1)}_1 = C_2(t) I_m(kr) \exp \left[ i (kz + m\phi) \right] \]  

(30)
\[
\psi_1^{(2)} = C_3(t)K_m(kr)\exp\left[i(kz + m\phi)\right]
\]

(31)

We have to apply the appropriate boundary conditions at \( r = R_0 \) to get \( C_2(t) \) and \( C_3(t) \):

\[
\psi_1 = \psi_2 \text{ and } n_0,\mathbf{E}^{(1)}_1 + n_1,\mathbf{E}^{(1)}_0 = n_0,\mathbf{E}^{(2)}_1 + n_1,\mathbf{E}^{(2)}_0
\]

from which, we get

\[
C_2(t) = C_3(t)\left(\frac{K_m(x)}{I_m(x)}\right), \quad x = kR_0
\]

(32)

\[
C_3(t) = \frac{-iE_0\left(\varepsilon^{(1)} - \varepsilon^{(2)}\right)I_m(x)}{\left(\varepsilon^{(1)}K_m(x)I'_m(x) - \varepsilon^{(2)}K'_m(x)I_m(x)\right)}
\]

(33)

Therefore,

\[
\mathbf{E}^{(1)}_1 = \nabla\left[\frac{-(ik)(-iE_0)\left(\varepsilon^{(1)} - \varepsilon^{(2)}\right)K_m(x)}{\left(\varepsilon^{(1)}K_m(x)I'_m(x) - \varepsilon^{(2)}K'_m(x)I_m(x)\right)}I_m(kr)\exp\left(i(kz + m\phi)\right)\right]
\]

(34)

\[
\mathbf{E}^{(2)}_1 = \nabla\left[\frac{-(ik)(-iE_0)\left(\varepsilon^{(1)} - \varepsilon^{(2)}\right)K_m(x)}{\left(\varepsilon^{(1)}K_m(x)I'_m(x) - \varepsilon^{(2)}K'_m(x)I_m(x)\right)}K_m(kr)\exp\left(i(kz + m\phi)\right)\right]
\]

(35)

By solving equations (18) and (19), we get

\[
V_1^{(1)} = B_1(t)I_m(kr)\exp (kz + m\phi)
\]

(36)

\[
V_1^{(2)} = B_2(t)K_m(kr)\exp (kz + m\phi)
\]

(37)

The coefficients \( B_1(t) \) and \( B_2(t) \) are determined upon using the boundary conditions:

\[
\begin{align*}
V_1^{(1)} + \eta \frac{\partial V_0^{(1)}}{\partial r} &= V_1^{(2)} + \eta \frac{\partial V_0^{(2)}}{\partial r} \\
\frac{\partial V_1^{(1)}}{\partial r} + \eta \frac{\partial^2 V_0^{(1)}}{\partial r^2} &= \frac{\partial V_1^{(2)}}{\partial r} + \eta \frac{\partial^2 V_0^{(2)}}{\partial r^2}
\end{align*}
\]

at \( r = R_0 \)

(38)

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and then upon utilizing the Wronskian relation (cf. Abramowitz and Stegun)

\[ I_m(x)K'_m(x) - I'_m(x)K_m(x) = -x^{-1} \]  

(39)

we get,

\[ V_1^{(1)} = 4\pi G\rho R_0\eta K_m(x)I_m(kr)\exp(i kz + m\phi) \]  

(40)

\[ V_1^{(2)} = 4\pi G\rho R_0\eta I_m(x)K_m(kr)\exp(i kz + m\phi) \]  

(41)

Now, we have to apply the continuity of the normal component of the total stress tensor at \( r = R_0 \):

\[ \Pi_1 + \rho \nu^{(1)} + \eta \frac{\partial}{\partial r} (\Pi_0^{(1)} + \rho \nu^{(1)}) + \frac{\varepsilon^{(1)}}{2} \left( 2E_0^{(1)}E_i^{(1)} \right) = \frac{\varepsilon^{(2)}}{2} \left( 2E_0^{(2)}E_i^{(2)} \right) \]  

(42)

with noting that \( \Pi_0 = \text{const} \).

We finally obtain, the second order integro-differential equation

\[ \frac{\partial^2 \eta}{\partial t^2} = -2ikU \cos \Omega t \frac{\partial \eta}{\partial t} + ikU\Omega \eta \sin \Omega t + k^2 U^2 \eta \cos \Omega t \]

\[ + 4\pi \rho G \frac{xI'_m(x)}{I_m(x)} \left[ I_m(x)K_m(x) - \frac{1}{2} \right] \eta \]

\[ - \frac{\varepsilon^{(1)}E_0^2}{\rho R_0^2} \eta \frac{\varepsilon^{(2)}}{\varepsilon^{(1)}} \left( 1 - \frac{\varepsilon^{(2)}}{\varepsilon^{(1)}} \right)^2 \left[ \frac{I'_m(x)K_m(x)}{I_m(x)K'_m(x)I_m(x)} \right] \eta \]  

(43)

4. DISCUSSION

Equation (43) is the second order integro-differential equation in the amplitude \( \eta(t) \) of the perturbation. It relates \( \eta \) with the modified Bessel functions \( I_m(x) \) and \( K_m(x) \), the wave numbers \( x \) and \( m \), the permittivity \( \varepsilon^{(1)} \) and \( \varepsilon^{(2)} \) of the fluid and tenuous medium and with the parameters \( \rho, R_0, E_0, V \) and \( \Omega \) of the problem. It contains \( \left( \frac{\varepsilon E_0^2}{\rho R_0^2} \right)^{1/2} \) as well as \( \left( 4\pi G\rho \right)^{1/2} \) each of unit \((\text{time})^{-1}\).

If we assume that \( \eta \equiv \exp(\sigma t) \) where \( \sigma \) is the growth rate of the perturbation, equation (43), yields
\[
\sigma^2 = (-2ikU \cos \Omega t) \sigma + ikU \Omega \sin \Omega t + k^2 U^2 \cos \Omega t
\]

\[
+ 4\pi G \rho \frac{xI_m'(x)}{I_m(x)} \left[ I_m(x)K_m(x) - \frac{1}{2} \right]
\]

\[
- \frac{\varepsilon^{(1)} E_0^2}{\rho R_0^2} \frac{x^2 \left( 1 - \frac{\varepsilon^{(2)}}{\varepsilon^{(1)}} \right)^2 I_m'(x)K_m(x)}{I_m'(x)K_m(x) - \frac{\varepsilon^{(2)}}{\varepsilon^{(1)}} K_m'(x)I_m(x)}
\]  

(44)

As \( t = 0 \) and the fluid streams uniformly \( u_0 = (0,0,U) \) equation (43) reduces to

\[
\left( \sigma + ikU \right)^2 = 4\pi G \rho \frac{xI_m'(x)}{I_m(x)} \left[ I_m(x)K_m(x) - \frac{1}{2} \right]
\]

\[
- \frac{\varepsilon^{(1)} E_0^2}{\rho R_0^2} \frac{x^2 \left( 1 - \frac{\varepsilon^{(2)}}{\varepsilon^{(1)}} \right)^2 I_m'(x)K_m(x)}{I_m'(x)K_m(x) - \frac{\varepsilon^{(2)}}{\varepsilon^{(1)}} K_m'(x)I_m(x)}
\]  

(45)

The relation (45), as \( U = 0 \) is derived by Radwan (1992) in examining the effect of the electric field on the self-gravitating fluid cylinder.

As \( u_0 = 0 \) and \( E_0 = 0 \), the relation (45) gives the self-gravitating dispersion relation of a fluid cylinder surrounded by self-gravitational tenuous medium, viz.,

\[
\sigma^2 = 4\pi G \rho \frac{xI_m'(x)}{I_m(x)} \left[ I_m(x)K_m(x) - \frac{1}{2} \right]
\]  

(46)

This relation has been early derived by Chandrasekhar (1981).

We know that (cf. Abramowitz and Stegun (1970))

\[
I_m(x) > 0, \quad K_m(x) > 0, \quad I_m'(x) > 0, \quad \text{and} \quad K_m'(x) < 0
\]  

(47)

In view of the inequalities (47), the analytical and numerical discussions of the relation (46) reveal to the following results.

When \( m = 0 \):

\[
\frac{\sigma^2}{4\pi \rho G} > 0 \quad \text{as} \quad 0 < x < 1.0667 \quad \text{while} \quad \frac{\sigma^2}{4\pi \rho G} \leq 0 \quad \text{as} \quad 1.0667 \leq x < \infty.
\]

When \( m \geq 0 \): we have \( 0 < x < \infty \).
This means that the model is self-gravitating unstable in the axisymmetric perturbation $m = 0$ in the domain $0 < x < 1.0667$. While it is stable in the neighboring axisymmetric domain $1.0667 \leq x < \infty$ and in the non-axisymmetric domain $0 < x < \infty$.

Therefore, we conclude that the fluid cylinder is self-gravitating unstable in small axisymmetric domain and stable otherwise.

As $U = 0$ and we neglect the self-gravitating force effect, the relation (45), reduces to

$$\sigma^2 = \frac{\varepsilon^{(1)} E_0^2}{\rho R_0^2} \frac{x^2 \left( 1 - \frac{\varepsilon^{(2)} I_m'(x) K_m(x)}{\varepsilon^{(1)} I_m(x) K_m(x)} \right)^2}{\varepsilon^{(2)} K_m'(x) I_m(x) - I_m'(x) K_m(x)}$$

(48)

The discussions of the relation (48), in view of the inequalities (47) show that the electrodynamic forces acting on the cylindrical fluid and the surrounding tenuous medium have strong stabilizing influence.

As $G = 0$ and $E_0 = 0$, the general relation (43), shows that the periodic oscillating stream has stabilizing tendency. However, the uniform stream is destabilizing. Therefore, in our present case the electrodynamic self-gravitational stable domains are enlarged and could suppress the unstable domains and consequently the model will be completely stable as the oscillating stream of the fluid is so strong.

REFERENCES